5 | RELATIVITY

Figure 5.1 Special relativity explains how time passes slightly differently on Earth and within the rapidly moving global positioning satellite (GPS). GPS units in vehicles could not find their correct location on Earth without taking this correction into account. (credit: modification of work by U.S. Air Force)

Chapter Outline

Introduction

The special theory of relativity was proposed in 1905 by Albert Einstein (1879–1955). It describes how time, space, and physical phenomena appear in different frames of reference that are moving at constant velocity with respect to each other. This differs from Einstein's later work on general relativity, which deals with any frame of reference, including accelerated frames.

The theory of relativity led to a profound change in the way we perceive space and time. The "common sense" rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. For example, the special theory of relativity tells us that measurements of length and time intervals are not the same in reference frames moving relative to one another. A particle might be observed to have a lifetime of 1.0×10^{-8} s in one reference frame, but a lifetime of 2.0×10^{-8} s in another; and an object might be measured to be 2.0 m long in one frame and 3.0 m long in another frame. These effects are usually significant only at speeds comparable to the speed of light, but even at the much lower speeds of the global positioning satellite, which requires extremely accurate time measurements to function, the different lengths of the same distance in different frames of reference are significant enough that they need to be taken into account.

Unlike Newtonian mechanics, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

The modifications of Newtonian mechanics in special relativity do not invalidate classical Newtonian mechanics or require its replacement. Instead, the equations of relativistic mechanics differ meaningfully from those of classical Newtonian mechanics only for objects moving at relativistic speeds (i.e., speeds less than, but comparable to, the speed of light). In the macroscopic world that you encounter in your daily life, the relativistic equations reduce to classical equations, and the predictions of classical Newtonian mechanics agree closely enough with experimental results to disregard relativistic corrections.

5.1 | Invariance of Physical Laws

Learning Objectives

By the end of this section, you will be able to:

- Describe the theoretical and experimental issues that Einstein's theory of special relativity addressed.
- State the two postulates of the special theory of relativity.

Suppose you calculate the hypotenuse of a right triangle given the base angles and adjacent sides. Whether you calculate the hypotenuse from one of the sides and the cosine of the base angle, or from the Pythagorean theorem, the results should agree. Predictions based on different principles of physics must also agree, whether we consider them principles of mechanics or principles of electromagnetism.

Albert Einstein pondered a disagreement between predictions based on electromagnetism and on assumptions made in classical mechanics. Specifically, suppose an observer measures the velocity of a light pulse in the observer's own **rest frame**; that is, in the frame of reference in which the observer is at rest. According to the assumptions long considered obvious in classical mechanics, if an observer measures a velocity $\vec{\mathbf{v}}$ in one frame of reference, and that frame of reference is moving with velocity $\vec{\mathbf{u}}$ past a second reference frame, an observer in the second frame measures the original velocity as $\vec{\mathbf{v}}' = \vec{\mathbf{v}} + \vec{\mathbf{u}}$. This sum of velocities is often referred to as **Galilean relativity**. If this principle is correct, the pulse of light that the observer measures as traveling with speed c travels at speed $c + u$ measured in the frame of the second observer. If we reasonably assume that the laws of electrodynamics are the same in both frames of reference, then the predicted speed of light (in vacuum) in both frames should be $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Each observer should measure the same speed of the light pulse with respect to that observer's own rest frame. To reconcile difficulties of this kind, Einstein constructed his **special theory of relativity**, which introduced radical new ideas about time and space that have since been confirmed experimentally.

Inertial Frames

All velocities are measured relative to some frame of reference. For example, a car's motion is measured relative to its starting position on the road it travels on; a projectile's motion is measured relative to the surface from which it is launched; and a planet's orbital motion is measured relative to the star it orbits. The frames of reference in which mechanics takes the simplest form are those that are not accelerating. Newton's first law, the law of inertia, holds exactly in such a frame.

Inertial Reference Frame

An **inertial frame of reference** is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.

For example, to a passenger inside a plane flying at constant speed and constant altitude, physics seems to work exactly the same as when the passenger is standing on the surface of Earth. When the plane is taking off, however, matters are somewhat more complicated. In this case, the passenger at rest inside the plane concludes that a net force F on an object is not equal to the product of mass and acceleration, ma. Instead, F is equal to ma plus a fictitious force. This situation is not as simple as in an inertial frame. The term "special" in "special relativity" refers to dealing only with inertial frames of reference. Einstein's later theory of general relativity deals with all kinds of reference frames, including accelerating, and therefore non-inertial, reference frames.

Einstein's First Postulate

Not only are the principles of classical mechanics simplest in inertial frames, but they are the same in all inertial frames. Einstein based the **first postulate** of his theory on the idea that this is true for all the laws of physics, not merely those in mechanics.

First Postulate of Special Relativity

The laws of physics are the same in all inertial frames of reference.

This postulate denies the existence of a special or preferred inertial frame. The laws of nature do not give us a way to endow any one inertial frame with special properties. For example, we cannot identify any inertial frame as being in a state of "absolute rest." We can only determine the relative motion of one frame with respect to another.

There is, however, more to this postulate than meets the eye. The laws of physics include only those that satisfy this postulate. We will see that the definitions of energy and momentum must be altered to fit this postulate. Another outcome of this postulate is the famous equation $E = mc^2$, which relates energy to mass.

Einstein's Second Postulate

The second postulate upon which Einstein based his theory of special relativity deals with the speed of light. Late in the nineteenth century, the major tenets of classical physics were well established. Two of the most important were the laws of electromagnetism and Newton's laws. Investigations such as Young's double-slit experiment in the early 1800s had convincingly demonstrated that light is a wave. Maxwell's equations of electromagnetism implied that electromagnetic waves travel at $c = 3.00\times10^8$ m/s in a vacuum, but they do not specify the frame of reference in which light has this speed. Many types of waves were known, and all travelled in some medium. Scientists therefore assumed that some medium carried the light, even in a vacuum, and that light travels at a speed c relative to that medium (often called "the aether").

Starting in the mid-1880s, the American physicist A.A. Michelson, later aided by E.W. Morley, made a series of direct measurements of the speed of light. They intended to deduce from their data the speed v at which Earth was moving through the mysterious medium for light waves. The speed of light measured on Earth should have been $c + v$ when Earth's motion was opposite to the medium's flow at speed u past the Earth, and $c - v$ when Earth was moving in the same direction as the medium. The results of their measurements were startling.

Michelson-Morley Experiment

The **Michelson-Morley experiment** demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the Sun.

The eventual conclusion derived from this result is that light, unlike mechanical waves such as sound, does not need a medium to carry it. Furthermore, the Michelson-Morley results implied that the speed of light c is independent of the motion of the source relative to the observer. That is, everyone observes light to move at speed c regardless of how they move relative to the light source or to one another. For several years, many scientists tried unsuccessfully to explain these results within the framework of Newton's laws.

In addition, there was a contradiction between the principles of electromagnetism and the assumption made in Newton's laws about relative velocity. Classically, the velocity of an object in one frame of reference and the velocity of that object in a second frame of reference relative to the first should combine like simple vectors to give the velocity seen in the second frame. If that were correct, then two observers moving at different speeds would see light traveling at different speeds. Imagine what a light wave would look like to a person traveling along with it (in vacuum) at a speed c. If such a motion were possible, then the wave would be stationary relative to the observer. It would have electric and magnetic fields whose strengths varied with position but were constant in time. This is not allowed by Maxwell's equations. So either Maxwell's equations are different in different inertial frames, or an object with mass cannot travel at speed c. Einstein concluded that the latter is true: An object with mass cannot travel at speed c. Maxwell's equations are correct, but Newton's addition of velocities is not correct for light.

Not until 1905, when Einstein published his first paper on special relativity, was the currently accepted conclusion reached. Based mostly on his analysis that the laws of electricity and magnetism would not allow another speed for light, and only slightly aware of the Michelson-Morley experiment, Einstein detailed his **second postulate of special relativity**.

Second Postulate of Special Relativity

Light travels in a vacuum with the same speed *c* in any direction in all inertial frames.

In other words, the speed of light has the same definite speed for any observer, regardless of the relative motion of the source. This deceptively simple and counterintuitive postulate, along with the first postulate, leave all else open for change. Among the changes are the loss of agreement on the time between events, the variation of distance with speed, and the realization that matter and energy can be converted into one another. We describe these concepts in the following sections.

5.1 Check Your Understanding Explain how special relativity differs from general relativity.

5.2 | Relativity of Simultaneity

Learning Objectives

By the end of this section, you will be able to:

- Show from Einstein's postulates that two events measured as simultaneous in one inertial frame are not necessarily simultaneous in all inertial frames.
- Describe how simultaneity is a relative concept for observers in different inertial frames in relative motion.

Do time intervals depend on who observes them? Intuitively, it seems that the time for a process, such as the elapsed time for a foot race (Figure 5.2), should be the same for all observers. In everyday experiences, disagreements over elapsed time have to do with the accuracy of measuring time. No one would be likely to argue that the actual time interval was different for the moving runner and for the stationary clock displayed. Carefully considering just how time is measured, however, shows that elapsed time does depends on the relative motion of an observer with respect to the process being measured.

Figure 5.2 Elapsed time for a foot race is the same for all observers, but at relativistic speeds, elapsed time depends on the motion of the observer relative to the location where the process being timed occurs. (credit: "Jason Edward Scott Bain"/Flickr)

Consider how we measure elapsed time. If we use a stopwatch, for example, how do we know when to start and stop the watch? One method is to use the arrival of light from the event. For example, if you're in a moving car and observe the light arriving from a traffic signal change from green to red, you know it's time to step on the brake pedal. The timing is more accurate if some sort of electronic detection is used, avoiding human reaction times and other complications.

Now suppose two observers use this method to measure the time interval between two flashes of light from flash lamps that are a distance apart (**Figure 5.3**). An observer *A* is seated midway on a rail car with two flash lamps at opposite sides equidistant from her. A pulse of light is emitted from each flash lamp and moves toward observer *A*, shown in frame (a) of the figure. The rail car is moving rapidly in the direction indicated by the velocity vector in the diagram. An observer *B* standing on the platform is facing the rail car as it passes and observes both flashes of light reach him simultaneously, as shown in frame (c). He measures the distances from where he saw the pulses originate, finds them equal, and concludes that the pulses were emitted simultaneously.

However, because of Observer *A*'s motion, the pulse from the right of the railcar, from the direction the car is moving, reaches her before the pulse from the left, as shown in frame (b). She also measures the distances from within her frame of reference, finds them equal, and concludes that the pulses were not emitted simultaneously.

The two observers reach conflicting conclusions about whether the two events at well-separated locations were simultaneous. Both frames of reference are valid, and both conclusions are valid. Whether two events at separate locations are simultaneous depends on the motion of the observer relative to the locations of the events.

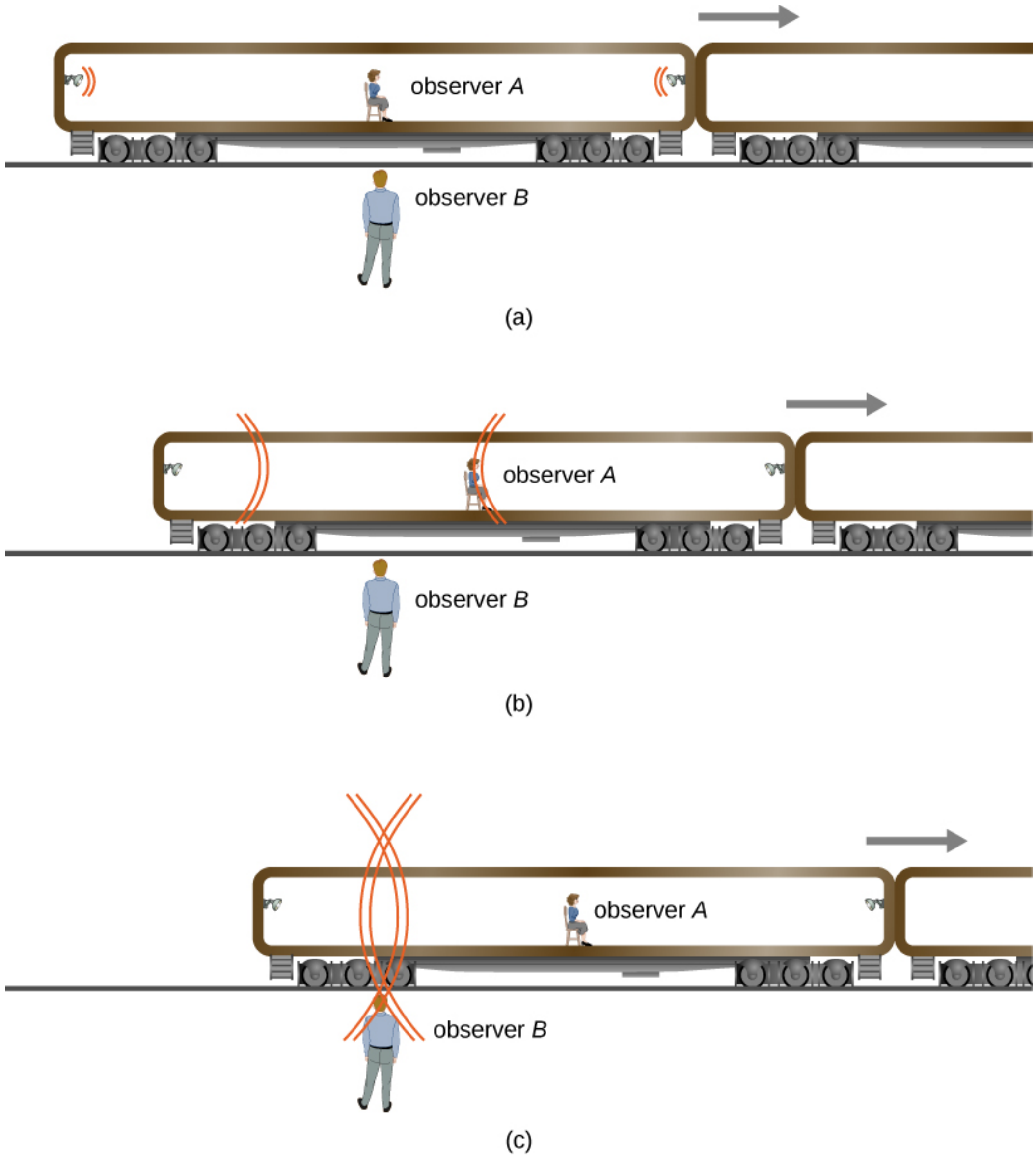


Figure 5.3 (a) Two pulses of light are emitted simultaneously relative to observer *B*. (c) The pulses reach observer *B*'s position simultaneously. (b) Because of *A*'s motion, she sees the pulse from the right first and concludes the bulbs did not flash simultaneously. Both conclusions are correct.

Here, the relative velocity between observers affects whether two events a distance apart are observed to be simultaneous. *Simultaneity is not absolute*. We might have guessed (incorrectly) that if light is emitted simultaneously, then two observers halfway between the sources would see the flashes simultaneously. But careful analysis shows this cannot be the case if the speed of light is the same in all inertial frames.

This type of *thought experiment* (in German, "Gedankenexperiment") shows that seemingly obvious conclusions must be changed to agree with the postulates of relativity. The validity of thought experiments can only be determined by actual observation, and careful experiments have repeatedly confirmed Einstein's theory of relativity.

5.3 | Time Dilation

Learning Objectives

By the end of this section, you will be able to:

- Explain how time intervals can be measured differently in different reference frames.
- Describe how to distinguish a proper time interval from a dilated time interval.
- Describe the significance of the muon experiment.
- Explain why the twin paradox is not a contradiction.
- Calculate time dilation given the speed of an object in a given frame.

The analysis of simultaneity shows that Einstein’s postulates imply an important effect: Time intervals have different values when measured in different inertial frames. Suppose, for example, an astronaut measures the time it takes for a pulse of light to travel a distance perpendicular to the direction of his ship’s motion (relative to an earthbound observer), bounce off a mirror, and return (**Figure 5.4**). How does the elapsed time that the astronaut measures in the spacecraft compare with the elapsed time that an earthbound observer measures by observing what is happening in the spacecraft?

Examining this question leads to a profound result. The elapsed time for a process depends on which observer is measuring it. In this case, the time measured by the astronaut (within the spaceship where the astronaut is at rest) is smaller than the time measured by the earthbound observer (to whom the astronaut is moving). The time elapsed for the same process is different for the observers, because the distance the light pulse travels in the astronaut’s frame is smaller than in the earthbound frame, as seen in **Figure 5.4**. Light travels at the same speed in each frame, so it takes more time to travel the greater distance in the earthbound frame.

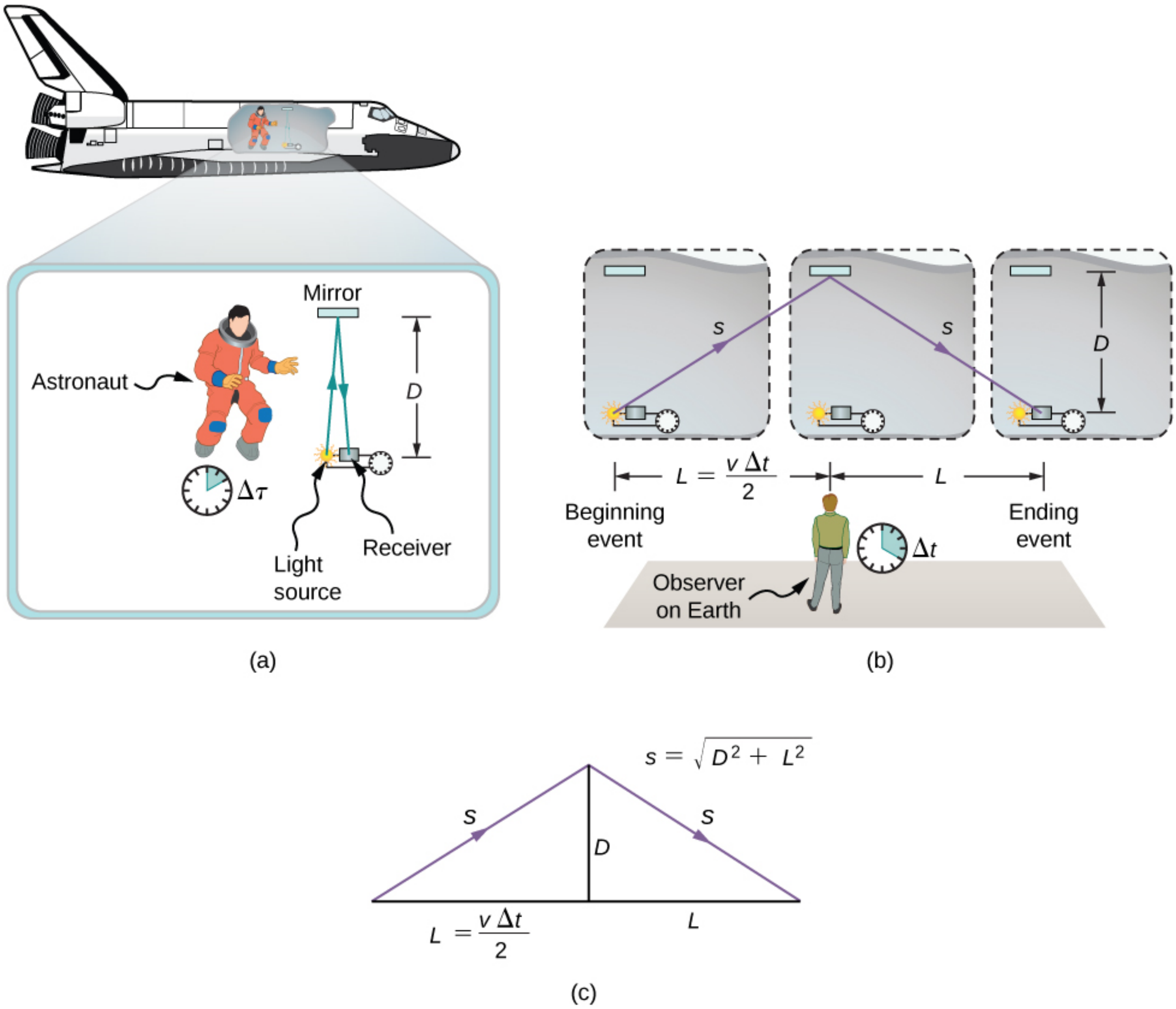


Figure 5.4 (a) An astronaut measures the time $\Delta\tau$ for light to travel distance $2D$ in the astronaut's frame. (b) A NASA scientist on Earth sees the light follow the longer path $2s$ and take a longer time Δt. (c) These triangles are used to find the relationship between the two distances D and s.

Time Dilation

Time dilation is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of the events (in which the events occur at the same location).

To quantitatively compare the time measurements in the two inertial frames, we can relate the distances in **Figure 5.4** to each other, then express each distance in terms of the time of travel (respectively either Δt or $\Delta\tau$) of the pulse in the corresponding reference frame. The resulting equation can then be solved for Δt in terms of $\Delta\tau$.

The lengths D and L in **Figure 5.4** are the sides of a right triangle with hypotenuse s. From the Pythagorean theorem,

$$s^2 = D^2 + L^2.$$

The lengths $2s$ and $2L$ are, respectively, the distances that the pulse of light and the spacecraft travel in time Δt in the earthbound observer's frame. The length D is the distance that the light pulse travels in time $\Delta\tau$ in the astronaut's frame. This gives us three equations:

$$2s = c\Delta t;\ 2L = v\Delta t;\ 2D = c\Delta\tau.$$

Note that we used Einstein's second postulate by taking the speed of light to be c in both inertial frames. We substitute these results into the previous expression from the Pythagorean theorem:

$$\begin{aligned} s^2 &= D^2 + L^2 \\ \left(c\frac{\Delta t}{2}\right)^2 &= \left(c\frac{\Delta \tau}{2}\right)^2 + \left(v\frac{\Delta t}{2}\right)^2. \end{aligned}$$

Then we rearrange to obtain

$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta\tau)^2.$$

Finally, solving for Δt in terms of $\Delta\tau$ gives us

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-(v/c)^2}}. \tag{5.1}$$

This is equivalent to

$$\Delta t = \gamma\Delta\tau,$$

where γ is the relativistic factor (often called the Lorentz factor) given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.2}$$

and v and c are the speeds of the moving observer and light, respectively.

Note the asymmetry between the two measurements. Only one of them is a measurement of the time interval between two events—the emission and arrival of the light pulse—at the same position. It is a measurement of the time interval in the rest frame of a single clock. The measurement in the earthbound frame involves comparing the time interval between two events that occur at different locations. The time interval between events that occur at a single location has a separate name to distinguish it from the time measured by the earthbound observer, and we use the separate symbol $\Delta\tau$ to refer to it throughout this chapter.

Proper Time

The **proper time** interval $\Delta\tau$ between two events is the time interval measured by an observer for whom both events occur at the same location.

The equation relating Δt and $\Delta\tau$ is truly remarkable. First, as stated earlier, elapsed time is not the same for different observers moving relative to one another, even though both are in inertial frames. A proper time interval $\Delta\tau$ for an observer who, like the astronaut, is moving with the apparatus, is smaller than the time interval for other observers. It is the smallest possible measured time between two events. The earthbound observer sees time intervals within the moving system as dilated (i.e., lengthened) relative to how the observer moving relative to Earth sees them within the moving system. Alternatively, according to the earthbound observer, less time passes between events within the moving frame. Note that the shortest elapsed time between events is in the inertial frame in which the observer sees the events (e.g., the emission and arrival of the light signal) occur at the same point.

This time effect is real and is not caused by inaccurate clocks or improper measurements. Time-interval measurements of the same event differ for observers in relative motion. The dilation of time is an intrinsic property of time itself. All clocks moving relative to an observer, including biological clocks, such as a person's heartbeat, or aging, are observed to run more slowly compared with a clock that is stationary relative to the observer.

Note that if the relative velocity is much less than the speed of light $(v << c)$, then v^2/c^2 is extremely small, and the elapsed times Δt and $\Delta \tau$ are nearly equal. At low velocities, physics based on modern relativity approaches classical physics—everyday experiences involve very small relativistic effects. However, for speeds near the speed of light, v^2/c^2 is close to one, so $\sqrt{1 - v^2/c^2}$ is very small and Δt becomes significantly larger than $\Delta \tau$.

Half-Life of a Muon

There is considerable experimental evidence that the equation $\Delta t = \gamma \Delta \tau$ is correct. One example is found in cosmic ray particles that continuously rain down on Earth from deep space. Some collisions of these particles with nuclei in the upper atmosphere result in short-lived particles called muons. The half-life (amount of time for half of a material to decay) of a muon is 1.52 μs when it is at rest relative to the observer who measures the half-life. This is the proper time interval $\Delta \tau$. This short time allows very few muons to reach Earth's surface and be detected if Newtonian assumptions about time and space were correct. However, muons produced by cosmic ray particles have a range of velocities, with some moving near the speed of light. It has been found that the muon's half-life as measured by an earthbound observer (Δt) varies with velocity exactly as predicted by the equation $\Delta t = \gamma \Delta \tau$. The faster the muon moves, the longer it lives. We on Earth see the muon last much longer than its half-life predicts within its own rest frame. As viewed from our frame, the muon decays more slowly than it does when at rest relative to us. A far larger fraction of muons reach the ground as a result.

Before we present the first example of solving a problem in relativity, we state a strategy you can use as a guideline for these calculations.

Problem-Solving Strategy: Relativity

1. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Look in particular for information on relative velocity v.
2. Identify exactly what needs to be determined in the problem (identify the unknowns).
3. Make certain you understand the conceptual aspects of the problem before making any calculations (express the answer as an equation). Decide, for example, which observer sees time dilated or length contracted before working with the equations or using them to carry out the calculation. If you have thought about who sees what, who is moving with the event being observed, who sees proper time, and so on, you will find it much easier to determine if your calculation is reasonable.
4. Determine the primary type of calculation to be done to find the unknowns identified above (do the calculation). You will find the section summary helpful in determining whether a length contraction, relativistic kinetic energy, or some other concept is involved.

Note *that you should not round off during the calculation*. As noted in the text, you must often perform your calculations to many digits to see the desired effect. You may round off at the very end of the problem solution, but do not use a rounded number in a subsequent calculation. Also, check the answer to see if it is reasonable: Does it make sense? This may be more difficult for relativity, which has few everyday examples to provide experience with what is reasonable. But you can look for velocities greater than c or relativistic effects that are in the wrong direction (such as a time contraction where a dilation was expected).

Example 5.1

Time Dilation in a High-Speed Vehicle

The Hypersonic Technology Vehicle 2 (HTV-2) is an experimental rocket vehicle capable of traveling at 21,000 km/h (5830 m/s). If an electronic clock in the HTV-2 measures a time interval of exactly 1-s duration, what would observers on Earth measure the time interval to be?

Strategy

Apply the time dilation formula to relate the proper time interval of the signal in HTV-2 to the time interval measured on the ground.

Solution

a. Identify the knowns: $\Delta\tau = 1\text{ s}; v = 5830\text{ m/s}$.

b. Identify the unknown: Δt.

c. Express the answer as an equation:

$$\Delta t = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

d. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \frac{1\text{ s}}{\sqrt{1 - \left(\frac{5830\text{ m/s}}{3.00 \times 10^8\text{ m/s}}\right)^2}} \\ &= 1.000000000189\text{ s} \\ &= 1\text{ s} + 1.89 \times 10^{-10}\text{ s}.\end{aligned}$$

Significance

The very high speed of the HTV-2 is still only 10^{-5} times the speed of light. Relativistic effects for the HTV-2 are negligible for almost all purposes, but are not zero.

Example 5.2

What Speeds are Relativistic?

How fast must a vehicle travel for 1 second of time measured on a passenger's watch in the vehicle to differ by 1% for an observer measuring it from the ground outside?

Strategy

Use the time dilation formula to find *v/c* for the given ratio of times.

Solution

a. Identify the known:

$$\frac{\Delta\tau}{\Delta t} = \frac{1}{1.01}.$$

b. Identify the unknown: *v/c*.

c. Express the answer as an equation:

$$\begin{aligned}\Delta t &= \gamma\Delta\tau = \frac{1}{\sqrt{1-v^2/c^2}}\Delta\tau \\ \frac{\Delta\tau}{\Delta t} &= \sqrt{1-v^2/c^2} \\ \left(\frac{\Delta\tau}{\Delta t}\right)^2 &= 1-\frac{v^2}{c^2} \\ \frac{v}{c} &= \sqrt{1-(\Delta\tau/\Delta t)^2}.\end{aligned}$$

d. Do the calculation:

$$\begin{aligned}\frac{v}{c} &= \sqrt{1-(1/1.01)^2} \\ &= 0.14.\end{aligned}$$

Significance

The result shows that an object must travel at very roughly 10% of the speed of light for its motion to produce significant relativistic time dilation effects.

Example 5.3

Calculating Δt for a Relativistic Event

Suppose a cosmic ray colliding with a nucleus in Earth's upper atmosphere produces a muon that has a velocity $v = 0.950c$. The muon then travels at constant velocity and lives 2.20 μs as measured in the muon's frame of reference. (You can imagine this as the muon's internal clock.) How long does the muon live as measured by an earthbound observer (**Figure 5.5**)?

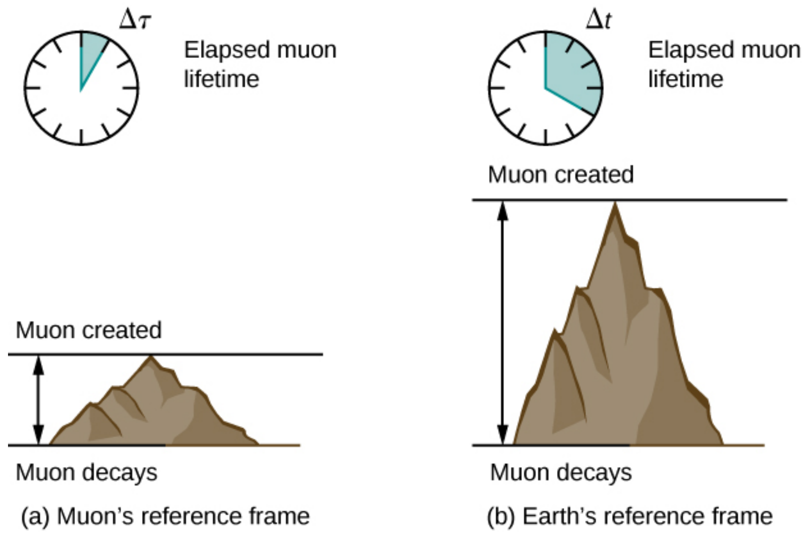


Figure 5.5 A muon in Earth's atmosphere lives longer as measured by an earthbound observer than as measured by the muon's internal clock.

As we will discuss later, in the muon's reference frame, it travels a shorter distance than measured in Earth's reference frame.

Strategy

A clock moving with the muon measures the proper time of its decay process, so the time we are given is $\Delta\tau = 2.20\mu s$. The earthbound observer measures Δt as given by the equation $\Delta t = \gamma\Delta\tau$. Because the velocity is given, we can calculate the time in Earth's frame of reference.

Solution

a. Identify the knowns: $v = 0.950c,\ \Delta\tau = 2.20\mu s$.

b. Identify the unknown: Δt.

c. Express the answer as an equation. Use:

$$\Delta t = \gamma \Delta\tau$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

d. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \gamma \Delta\tau \\ &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\Delta\tau \\ &= \frac{2.20\mu s}{\sqrt{1 - (0.950)^2}} \\ &= 7.05\ \mu s.\end{aligned}$$

Remember to keep extra significant figures until the final answer.

Significance

One implication of this example is that because $\gamma = 3.20$ at 95.0% of the speed of light $(v = 0.950c)$, the relativistic effects are significant. The two time intervals differ by a factor of 3.20, when classically they would be the same. Something moving at 0.950*c* is said to be highly relativistic.

Example 5.4

Relativistic Television

A non-flat screen, older-style television display (**Figure 5.6**) works by accelerating electrons over a short distance to relativistic speed, and then using electromagnetic fields to control where the electron beam strikes a fluorescent layer at the front of the tube. Suppose the electrons travel at 6.00×10^7 m/s through a distance of 0.200 m from the start of the beam to the screen. (a) What is the time of travel of an electron in the rest frame of the television set? (b) What is the electron's time of travel in its own rest frame?

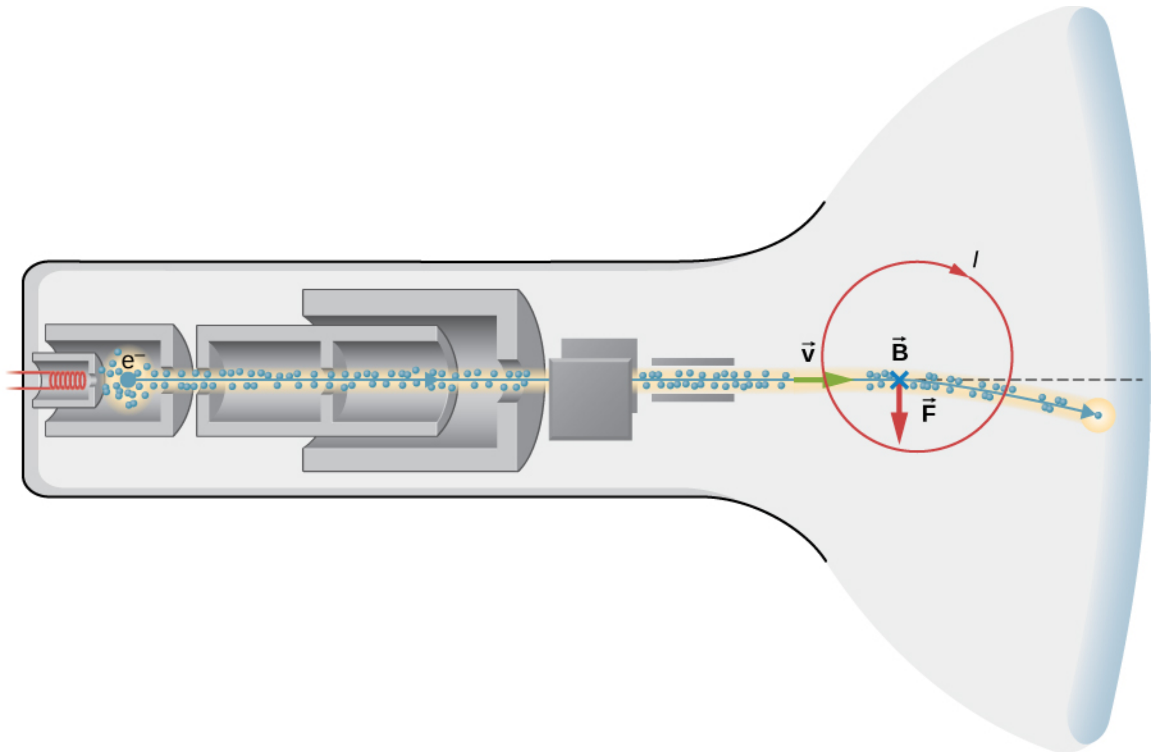


Figure 5.6 The electron beam in a cathode ray tube television display.

Strategy for (a)

(a) Calculate the time from $vt = d$. Even though the speed is relativistic, the calculation is entirely in one frame of reference, and relativity is therefore not involved.

Solution

a. Identify the knowns:

$$v = 6.00 \times 10^7 \text{ m/s}; d = 0.200 \text{ m}.$$

b. Identify the unknown: the time of travel Δt.

c. Express the answer as an equation:

$$\Delta t = \frac{d}{v}.$$

d. Do the calculation:

$$\begin{aligned} t &= \frac{0.200 \text{ m}}{6.00 \times 10^7 \text{ m/s}} \\ &= 3.33 \times 10^{-9} \text{ s}. \end{aligned}$$

Significance

The time of travel is extremely short, as expected. Because the calculation is entirely within a single frame of reference, relativity is not involved, even though the electron speed is close to c.

Strategy for (b)

(b) In the frame of reference of the electron, the vacuum tube is moving and the electron is stationary. The electron-emitting cathode leaves the electron and the front of the vacuum tube strikes the electron with the electron at the same location. Therefore we use the time dilation formula to relate the proper time in the electron rest frame to the time in the television frame.

Solution

a. Identify the knowns (from part a):

$$\Delta t = 3.33 \times 10^{-9}\ \text{s};\ v = 6.00 \times 10^{7}\ \text{m/s};\ d = 0.200\ \text{m}.$$

b. Identify the unknown: τ.

c. Express the answer as an equation:

$$\begin{aligned}\Delta t &= \gamma \Delta \tau = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}} \\ \Delta \tau &= \Delta t \sqrt{1 - v^2/c^2}.\end{aligned}$$

d. Do the calculation:

$$\begin{aligned}\Delta \tau &= \left(3.33 \times 10^{-9}\ \text{s}\right)\sqrt{1 - \left(\frac{6.00 \times 10^{7}\ \text{m/s}}{3.00 \times 10^{8}\ \text{m/s}}\right)^2} \\ &= 3.26 \times 10^{-9}\ \text{s}.\end{aligned}$$

Significance

The time of travel is shorter in the electron frame of reference. Because the problem requires finding the time interval measured in different reference frames for the same process, relativity is involved. If we had tried to calculate the time in the electron rest frame by simply dividing the 0.200 m by the speed, the result would be slightly incorrect because of the relativistic speed of the electron.

5.2 Check Your Understanding What is γ if $v = 0.650c$?

The Twin Paradox

An intriguing consequence of time dilation is that a space traveler moving at a high velocity relative to Earth would age less than the astronaut's earthbound twin. This is often known as the twin paradox. Imagine the astronaut moving at such a velocity that $\gamma = 30.0$, as in **Figure 5.7**. A trip that takes 2.00 years in her frame would take 60.0 years in the earthbound twin's frame. Suppose the astronaut travels 1.00 year to another star system, briefly explores the area, and then travels 1.00 year back. An astronaut who was 40 years old at the start of the trip would be would be 42 when the spaceship returns. Everything on Earth, however, would have aged 60.0 years. The earthbound twin, if still alive, would be 100 years old.

The situation would seem different to the astronaut in **Figure 5.7**. Because motion is relative, the spaceship would seem to be stationary and Earth would appear to move. (This is the sensation you have when flying in a jet.) Looking out the window of the spaceship, the astronaut would see time slow down on Earth by a factor of $\gamma = 30.0$. Seen from the spaceship, the earthbound sibling will have aged only 2/30, or 0.07, of a year, whereas the astronaut would have aged 2.00 years.

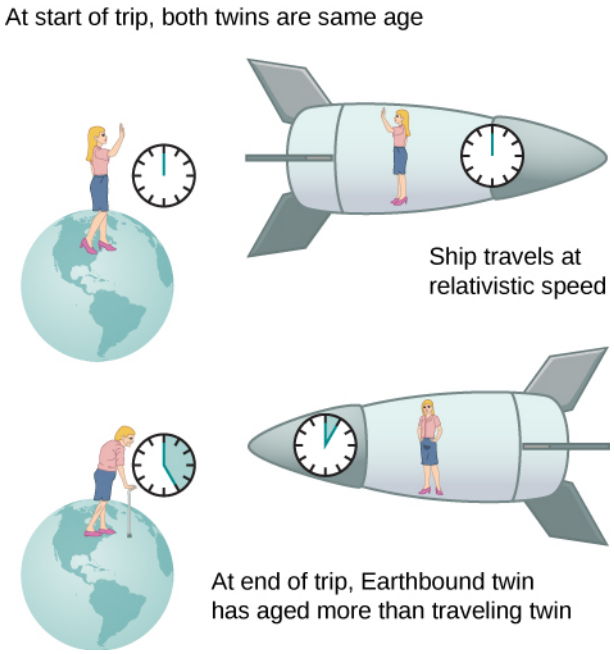


Figure 5.7 The twin paradox consists of the conflicting conclusions about which twin ages more as a result of a long space journey at relativistic speed.

The paradox here is that the two twins cannot both be correct. As with all paradoxes, conflicting conclusions come from a false premise. In fact, the astronaut's motion is significantly different from that of the earthbound twin. The astronaut accelerates to a high velocity and then decelerates to view the star system. To return to Earth, she again accelerates and decelerates. The spacecraft is not in a single inertial frame to which the time dilation formula can be directly applied. That is, the astronaut twin changes inertial references. The earthbound twin does not experience these accelerations and remains in the same inertial frame. Thus, the situation is not symmetric, and it is incorrect to claim that the astronaut observes the same effects as her twin. The lack of symmetry between the twins will be still more evident when we analyze the journey later in this chapter in terms of the path the astronaut follows through four-dimensional space-time.

In 1971, American physicists Joseph Hafele and Richard Keating verified time dilation at low relative velocities by flying extremely accurate atomic clocks around the world on commercial aircraft. They measured elapsed time to an accuracy of a few nanoseconds and compared it with the time measured by clocks left behind. Hafele and Keating's results were within experimental uncertainties of the predictions of relativity. Both special and general relativity had to be taken into account, because gravity and accelerations were involved as well as relative motion.

5.3 **Check Your Understanding** a. A particle travels at $1.90 \times 10^8\,\text{m/s}$ and lives $2.10 \times 10^{-8}\,\text{s}$ when at rest relative to an observer. How long does the particle live as viewed in the laboratory?

b. Spacecraft *A* and *B* pass in opposite directions at a relative speed of $4.00 \times 10^7\,\text{m/s}$. An internal clock in spacecraft *A* causes it to emit a radio signal for 1.00 s. The computer in spacecraft *B* corrects for the beginning and end of the signal having traveled different distances, to calculate the time interval during which ship *A* was emitting the signal. What is the time interval that the computer in spacecraft *B* calculates?

5.4 | Length Contraction

Learning Objectives

By the end of this section, you will be able to:

- Explain how simultaneity and length contraction are related.
- Describe the relation between length contraction and time dilation and use it to derive the length-contraction equation.

The length of the train car in **Figure 5.8** is the same for all the passengers. All of them would agree on the simultaneous location of the two ends of the car and obtain the same result for the distance between them. But simultaneous events in one inertial frame need not be simultaneous in another. If the train could travel at relativistic speeds, an observer on the ground would see the simultaneous locations of the two endpoints of the car at a different distance apart than observers inside the car. Measured distances need not be the same for different observers when relativistic speeds are involved.

Figure 5.8 People might describe distances differently, but at relativistic speeds, the distances really are different. (credit: “russavia”/Flickr)

Proper Length

Two observers passing each other always see the same value of their relative speed. Even though time dilation implies that the train passenger and the observer standing alongside the tracks measure different times for the train to pass, they still agree that relative speed, which is distance divided by elapsed time, is the same. If an observer on the ground and one on the train measure a different time for the length of the train to pass the ground observer, agreeing on their relative speed means they must also see different distances traveled.

The muon discussed in **Example 5.3** illustrates this concept (**Figure 5.9**). To an observer on Earth, the muon travels at $0.950c$ for 7.05 μs from the time it is produced until it decays. Therefore, it travels a distance relative to Earth of:

$$L_0 = v\Delta t = (0.950)(3.00 \times 10^8 \text{ m/s})(7.05 \times 10^{-6} \text{ s}\big) = 2.01 \text{ km}.$$

In the muon frame, the lifetime of the muon is 2.20 μs. In this frame of reference, the Earth, air, and ground have only enough time to travel:

$$L = v\Delta\tau = (0.950)(3.00 \times 10^8 \text{ m/s})(2.20 \times 10^{-6} \text{ s}) \text{ km} = 0.627 \text{ km}.$$

The distance between the same two events (production and decay of a muon) depends on who measures it and how they are moving relative to it.

Proper Length

Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.

The earthbound observer measures the proper length L_0 because the points at which the muon is produced and decays are stationary relative to Earth. To the muon, Earth, air, and clouds are moving, so the distance L it sees is not the proper length.

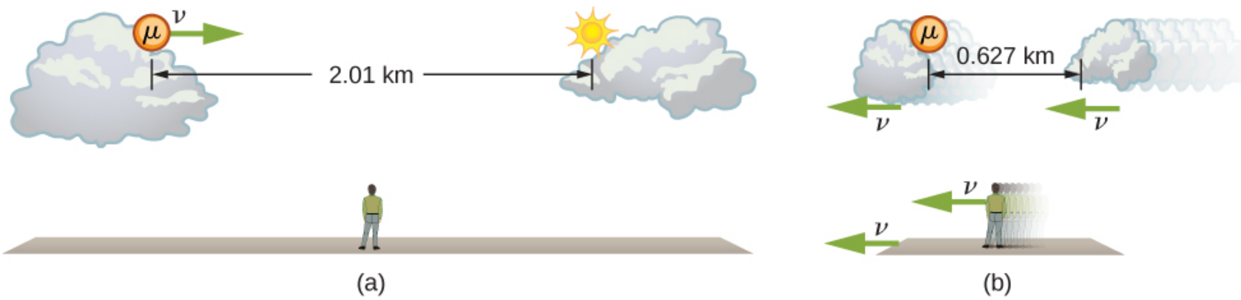


Figure 5.9 (a) The earthbound observer sees the muon travel 2.01 km. (b) The same path has length 0.627 km seen from the muon's frame of reference. The Earth, air, and clouds are moving relative to the muon in its frame, and have smaller lengths along the direction of travel.

Length Contraction

To relate distances measured by different observers, note that the velocity relative to the earthbound observer in our muon example is given by

$$v = \frac{L_0}{\Delta t}.$$

The time relative to the earthbound observer is $\Delta t,$ because the object being timed is moving relative to this observer. The velocity relative to the moving observer is given by

$$v = \frac{L}{\Delta \tau}.$$

The moving observer travels with the muon and therefore observes the proper time $\Delta \tau.$ The two velocities are identical; thus,

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta \tau}.$$

We know that $\Delta t = \gamma \Delta \tau.$ Substituting this equation into the relationship above gives

$$L = \frac{L_0}{\gamma}. \tag{5.3}$$

Substituting for γ gives an equation relating the distances measured by different observers.

Length Contraction

Length contraction is the decrease in the measured length of an object from its proper length when measured in a reference frame that is moving with respect to the object:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{5.4}$$

where L_0 is the length of the object in its rest frame, and L is the length in the frame moving with velocity v.

If we measure the length of anything moving relative to our frame, we find its length L to be smaller than the proper length

L_0 that would be measured if the object were stationary. For example, in the muon's rest frame, the distance Earth moves between where the muon was produced and where it decayed is shorter than the distance traveled as seen from the Earth's frame. Those points are fixed relative to Earth but are moving relative to the muon. Clouds and other objects are also contracted along the direction of motion as seen from muon's rest frame.

Thus, two observers measure different distances along their direction of relative motion, depending on which one is measuring distances between objects at rest.

But what about distances measured in a direction perpendicular to the relative motion? Imagine two observers moving along their *x*-axes and passing each other while holding meter sticks vertically in the *y*-direction. **Figure 5.10** shows two meter sticks M and M' that are at rest in the reference frames of two boys S and S', respectively. A small paintbrush is attached to the top (the 100-cm mark) of stick M'. Suppose that S' is moving to the right at a very high speed *v* relative to S, and the sticks are oriented so that they are perpendicular, or transverse, to their relative velocity vector. The sticks are held so that as they pass each other, their lower ends (the 0-cm marks) coincide. Assume that when S looks at his stick M afterwards, he finds a line painted on it, just below the top of the stick. Because the brush is attached to the top of the other boy's stick M', S can only conclude that stick M' is less than 1.0 m long.

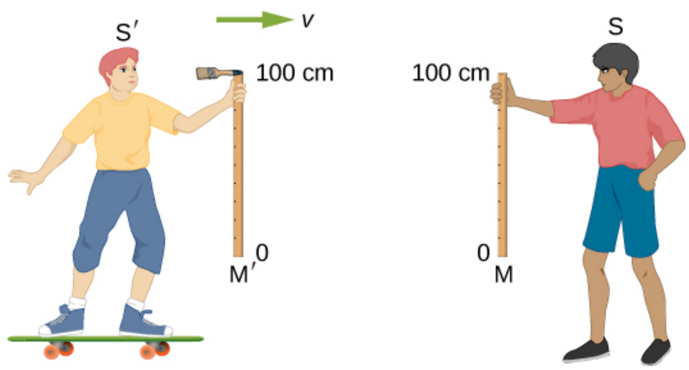


Figure 5.10 Meter sticks M and M' are stationary in the reference frames of observers S and S', respectively. As the sticks pass, a small brush attached to the 100-cm mark of M' paints a line on M.

Now when the boys approach each other, S', like S, sees a meter stick moving toward him with speed *v*. Because their situations are symmetric, each boy must make the same measurement of the stick in the other frame. So, if S measures stick M' to be less than 1.0 m long, S' must measure stick M to be also less than 1.0 m long, and S' must see his paintbrush pass over the top of stick M and not paint a line on it. In other words, after the same event, one boy sees a painted line on a stick, while the other does not see such a line on that same stick!

Einstein's first postulate requires that the laws of physics (as, for example, applied to painting) predict that S and S', who are both in inertial frames, make the same observations; that is, S and S' must either both see a line painted on stick M, or both not see that line. We are therefore forced to conclude our original assumption that S saw a line painted below the top of his stick was wrong! Instead, S finds the line painted right at the 100-cm mark on M. Then both boys will agree that a line is painted on M, and they will also agree that both sticks are exactly 1 m long. We conclude then that measurements of a transverse *length must be the same in different inertial frames*.

Example 5.5

Calculating Length Contraction

Suppose an astronaut, such as the twin in the twin paradox discussion, travels so fast that $\gamma = 30.00$. (a) The astronaut travels from Earth to the nearest star system, Alpha Centauri, 4.300 light years (ly) away as measured by an earthbound observer. How far apart are Earth and Alpha Centauri as measured by the astronaut? (b) In terms of *c*, what is the astronaut's velocity relative to Earth? You may neglect the motion of Earth relative to the sun (**Figure 5.11**).

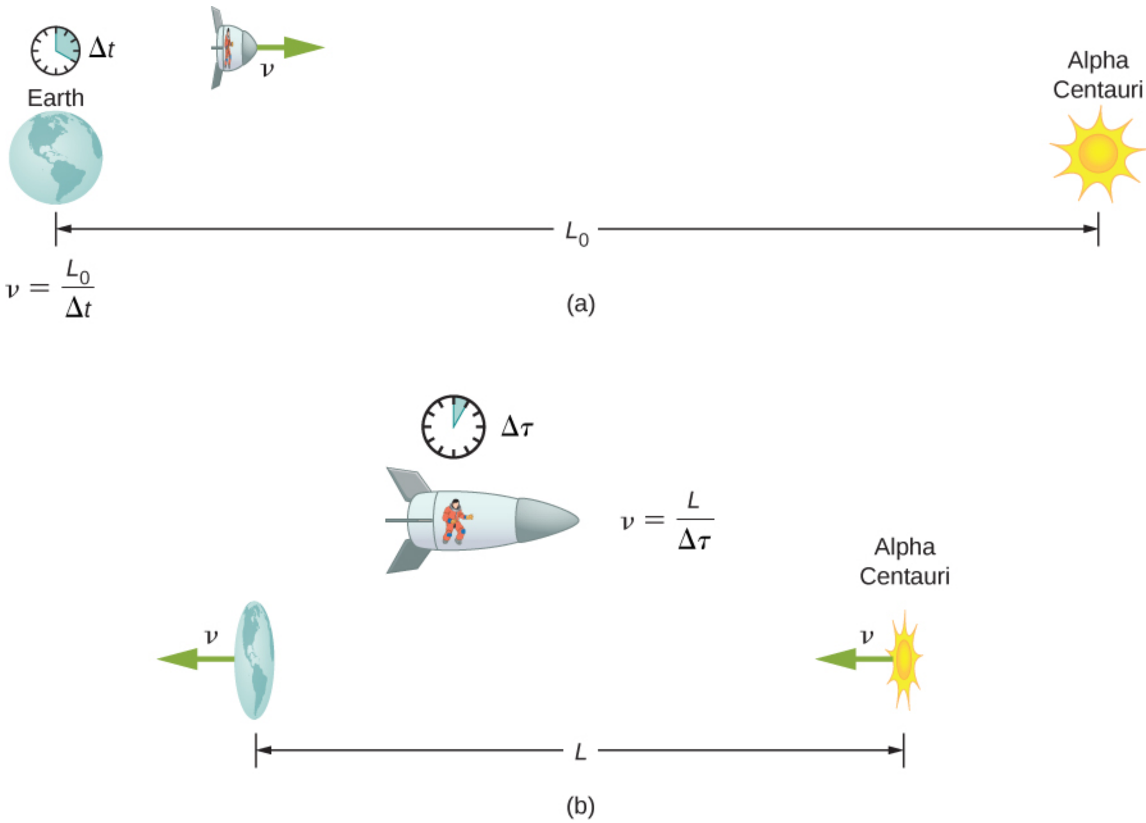


Figure 5.11 (a) The earthbound observer measures the proper distance between Earth and Alpha Centauri. (b) The astronaut observes a length contraction because Earth and Alpha Centauri move relative to her ship. She can travel this shorter distance in a smaller time (her proper time) without exceeding the speed of light.

Strategy

First, note that a light year (ly) is a convenient unit of distance on an astronomical scale—it is the distance light travels in a year. For part (a), the 4.300-ly distance between Alpha Centauri and Earth is the proper distance L_0, because it is measured by an earthbound observer to whom both stars are (approximately) stationary. To the astronaut, Earth and Alpha Centauri are moving past at the same velocity, so the distance between them is the contracted length L. In part (b), we are given γ, so we can find v by rearranging the definition of γ to express v in terms of c.

Solution for (a)

For part (a):

a. Identify the knowns: $L_0 = 4.300 \text{ ly}; \gamma = 30.00.$

b. Identify the unknown: L.

c. Express the answer as an equation: $L = \frac{L_0}{\gamma}.$

d. Do the calculation:

$$\begin{aligned} L &= \frac{L_0}{\gamma} \\ &= \frac{4.300 \text{ ly}}{30.00} \\ &= 0.1433 \text{ ly}. \end{aligned}$$

Solution for (b)

For part (b):

a. Identify the known: $\gamma = 30.00$.

b. Identify the unknown: v in terms of c.

c. Express the answer as an equation. Start with:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then solve for the unknown v/c by first squaring both sides and then rearranging:

$$\begin{aligned} \gamma^2 &= \frac{1}{1 - \frac{v^2}{c^2}} \\ \frac{v^2}{c^2} &= 1 - \frac{1}{\gamma^2} \\ \frac{v}{c} &= \sqrt{1 - \frac{1}{\gamma^2}}. \end{aligned}$$

d. Do the calculation:

$$\begin{aligned} \frac{v}{c} &= \sqrt{1 - \frac{1}{\gamma^2}} \\ &= \sqrt{1 - \frac{1}{(30.00)^2}} \\ &= 0.99944 \end{aligned}$$

or

$$v = 0.9994\,c.$$

Significance

Remember not to round off calculations until the final answer, or you could get erroneous results. This is especially true for special relativity calculations, where the differences might only be revealed after several decimal places. The relativistic effect is large here $(\gamma = 30.00)$, and we see that v is approaching (not equaling) the speed of light. Because the distance as measured by the astronaut is so much smaller, the astronaut can travel it in much less time in her frame.

People traveling at extremely high velocities could cover very large distances (thousands or even millions of light years) and age only a few years on the way. However, like emigrants in past centuries who left their home, these people would leave the Earth they know forever. Even if they returned, thousands to millions of years would have passed on Earth, obliterating most of what now exists. There is also a more serious practical obstacle to traveling at such velocities; immensely greater energies would be needed to achieve such high velocities than classical physics predicts can be attained. This will be discussed later in the chapter.

Why don't we notice length contraction in everyday life? The distance to the grocery store does not seem to depend on whether we are moving or not. Examining the equation $L = L_0\sqrt{1 - \frac{v^2}{c^2}}$, we see that at low velocities $(v \ll c)$, the lengths are nearly equal, which is the classical expectation. But length contraction is real, if not commonly experienced. For example, a charged particle such as an electron traveling at relativistic velocity has electric field lines that are compressed along the direction of motion as seen by a stationary observer (**Figure 5.12**). As the electron passes a detector, such as a coil of wire, its field interacts much more briefly, an effect observed at particle accelerators such as the 3-km-long Stanford Linear Accelerator (SLAC). In fact, to an electron traveling down the beam pipe at SLAC, the accelerator and Earth are all moving by and are length contracted. The relativistic effect is so great that the accelerator is only 0.5 m long to the electron. It is actually easier to get the electron beam down the pipe, because the beam does not have to be as precisely aimed to get down a short pipe as it would to get down a pipe 3 km long. This, again, is an experimental verification of the special theory of relativity.

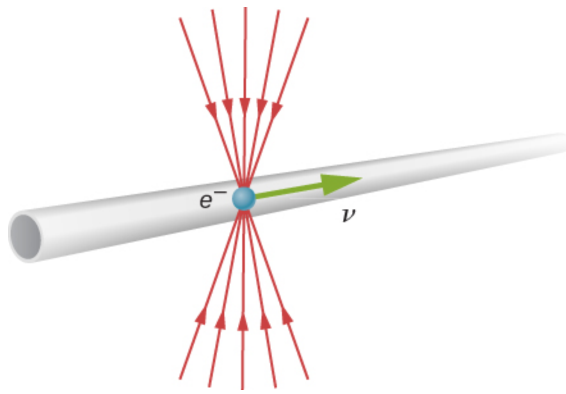


Figure 5.12 The electric field lines of a high-velocity charged particle are compressed along the direction of motion by length contraction, producing an observably different signal as the particle goes through a coil.

5.4 **Check Your Understanding** A particle is traveling through Earth's atmosphere at a speed of $0.750c$. To an earthbound observer, the distance it travels is 2.50 km. How far does the particle travel as viewed from the particle's reference frame?

5.5 | The Lorentz Transformation

Learning Objectives

- Describe the Galilean transformation of classical mechanics, relating the position, time, velocities, and accelerations measured in different inertial frames
- Derive the corresponding Lorentz transformation equations, which, in contrast to the Galilean transformation, are consistent with special relativity
- Explain the Lorentz transformation and many of the features of relativity in terms of four-dimensional space-time

We have used the postulates of relativity to examine, in particular examples, how observers in different frames of reference measure different values for lengths and the time intervals. We can gain further insight into how the postulates of relativity change the Newtonian view of time and space by examining the transformation equations that give the space and time coordinates of events in one inertial reference frame in terms of those in another. We first examine how position and time coordinates transform between inertial frames according to the view in Newtonian physics. Then we examine how this has to be changed to agree with the postulates of relativity. Finally, we examine the resulting Lorentz transformation equations and some of their consequences in terms of four-dimensional space-time diagrams, to support the view that the consequences of special relativity result from the properties of time and space itself, rather than electromagnetism.

The Galilean Transformation Equations

An **event** is specified by its location and time (x, y, z, t) relative to one particular inertial frame of reference S. As an example, (x, y, z, t) could denote the position of a particle at time t, and we could be looking at these positions for many different times to follow the motion of the particle. Suppose a second frame of reference S' moves with velocity v with respect to the first. For simplicity, assume this relative velocity is along the x-axis. The relation between the time and coordinates in the two frames of reference is then

$$x = x' + vt, \quad y = y', \quad z = z'.$$

Implicit in these equations is the assumption that time measurements made by observers in both S and S' are the same. That is,

$$t = t'.$$

These four equations are known collectively as the **Galilean transformation**.

We can obtain the Galilean velocity and acceleration transformation equations by differentiating these equations with respect to time. We use *u* for the velocity of a particle throughout this chapter to distinguish it from *v*, the relative velocity of two reference frames. Note that, for the Galilean transformation, the increment of time used in differentiating to calculate the particle velocity is the same in both frames, $dt = dt'$. Differentiation yields

$$u_x = u'_x + v, \quad u_y = u'_y, \quad u_z = u'_z$$

and

$$a_x = a'_x, \quad a_y = a'_y, \quad a_z = a'_z.$$

We denote the velocity of the particle by *u* rather than *v* to avoid confusion with the velocity *v* of one frame of reference with respect to the other. Velocities in each frame differ by the velocity that one frame has as seen from the other frame. Observers in both frames of reference measure the same value of the acceleration. Because the mass is unchanged by the transformation, and distances between points are uncharged, observers in both frames see the same forces $F = ma$ acting between objects and the same form of Newton's second and third laws in all inertial frames. The laws of mechanics are consistent with the first postulate of relativity.

The Lorentz Transformation Equations

The Galilean transformation nevertheless violates Einstein's postulates, because the velocity equations state that a pulse of light moving with speed *c* along the *x*-axis would travel at speed $c - v$ in the other inertial frame. Specifically, the spherical pulse has radius $r = ct$ at time *t* in the unprimed frame, and also has radius $r' = ct'$ at time t' in the primed frame. Expressing these relations in Cartesian coordinates gives

$$\begin{aligned} x^2 + y^2 + z^2 - c^2t^2 &= 0 \\ x'^2 + y'^2 + z'^2 - c^2t'^2 &= 0. \end{aligned}$$

The left-hand sides of the two expressions can be set equal because both are zero. Because $y = y'$ and $z = z'$, we obtain

$$x^2 - c^2t^2 = x'^2 - c^2t'^2. \tag{5.5}$$

This cannot be satisfied for nonzero relative velocity *v* of the two frames if we assume the Galilean transformation results in $t = t'$ with $x = x' + vt'$.

To find the correct set of transformation equations, assume the two coordinate systems *S* and S' in **Figure 5.13**. First suppose that an event occurs at $(x', 0, 0, t')$ in S' and at $(x, 0, 0, t)$ in *S*, as depicted in the figure.

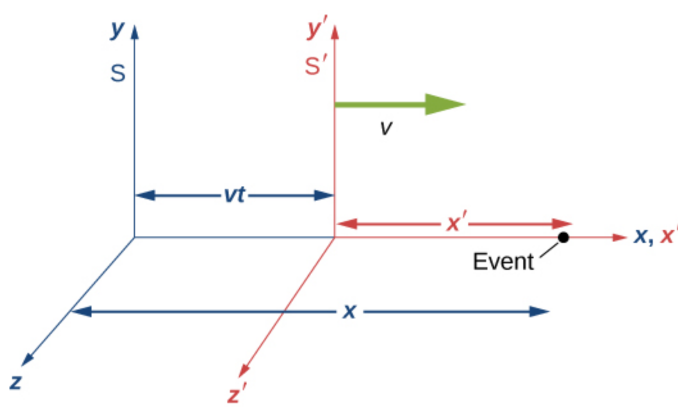


Figure 5.13 An event occurs at (*x*, 0, 0, *t*) in *S* and at $(x', 0, 0, t')$ in S'. The Lorentz transformation equations relate events in the two systems.

Suppose that at the instant that the origins of the coordinate systems in *S* and S' coincide, a flash bulb emits a spherically spreading pulse of light starting from the origin. At time *t*, an observer in *S* finds the origin of S' to be at $x = vt$. With the help of a friend in *S*, the S' observer also measures the distance from the event to the origin of S' and finds it to be

$x'\sqrt{1 - v^2/c^2}$. This follows because we have already shown the postulates of relativity to imply length contraction. Thus the position of the event in S is

$$x = vt + x'\sqrt{1 - v^2/c^2}$$

and

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}.$$

The postulates of relativity imply that the equation relating distance and time of the spherical wave front:

$$x^2 + y^2 + z^2 - c^2 t^2 = 0$$

must apply both in terms of primed and unprimed coordinates, which was shown above to lead to **Equation 5.5**:

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2.$$

We combine this with the equation relating x and x' to obtain the relation between t and t' :

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$

The equations relating the time and position of the events as seen in S are then

$$\begin{aligned} t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \\ x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z'. \end{aligned}$$

This set of equations, relating the position and time in the two inertial frames, is known as the **Lorentz transformation**. They are named in honor of H.A. Lorentz (1853–1928), who first proposed them. Interestingly, he justified the transformation on what was eventually discovered to be a fallacious hypothesis. The correct theoretical basis is Einstein's special theory of relativity.

The reverse transformation expresses the variables in S in terms of those in S'. Simply interchanging the primed and unprimed variables and substituting gives:

$$\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z. \end{aligned}$$

Example 5.6

Using the Lorentz Transformation for Time

Spacecraft S' is on its way to Alpha Centauri when Spacecraft S passes it at relative speed $c/2$. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship S.

Solution

a. Identify the known: $\Delta t' = t_2' - t_1' = 1.2\text{ s}$; $\Delta x' = x'_2 - x'_1 = 0$.

b. Identify the unknown: $\Delta t = t_2 - t_1$.

c. Express the answer as an equation. The time signal starts as (x', t_1') and stops at (x', t_2'). Note that the x' coordinate of both events is the same because the clock is at rest in S'. Write the first Lorentz transformation equation in terms of $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and similarly for the primed coordinates, as:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Because the position of the clock in S' is fixed, $\Delta x' = 0$, and the time interval Δt becomes:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

d. Do the calculation.
With $\Delta t' = 1.2\ \text{s}$ this gives:

$$\Delta t = \frac{1.2\ \text{s}}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} = 1.6\ \text{s}.$$

Note that the Lorentz transformation reproduces the time dilation equation.

Example 5.7

Using the Lorentz Transformation for Length

A surveyor measures a street to be $L = 100\ \text{m}$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed 0.20*c*, assuming the *x* coordinates of the two frames coincide at time $t = 0$.

Solution

a. Identify the known: $L = 100\ \text{m}; v = 0.20c; \Delta\tau = 0.$

b. Identify the unknown: L'.

c. Express the answer as an equation. The surveyor in frame S has measured the two ends of the stick simultaneously, and found them at rest at x_2 and x_1 a distance $L = x_2 - x_1 = 100\ \text{m}$ apart. The spaceship crew measures the simultaneous location of the ends of the sticks in their frame. To relate the lengths recorded by observers in S' and S, respectively, write the second of the four Lorentz transformation equations as:

$$\begin{aligned} x'_2 - x'_1 &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

d. Do the calculation. Because $x'_2 - x'_1 = 100\ \text{m}$, the length of the moving stick is equal to:

$$\begin{aligned} L' &= (100\ \text{m})\sqrt{1 - v^2/c^2} \\ &= (100\ \text{m})\sqrt{1 - (0.20)^2} \\ &= 98.0\ \text{m}. \end{aligned}$$

Note that the Lorentz transformation gave the length contraction equation for the street.

Example 5.8

Lorentz Transformation and Simultaneity

The observer shown in **Figure 5.14** standing by the railroad tracks sees the two bulbs flash simultaneously at both ends of the 26 m long passenger car when the middle of the car passes him at a speed of $c/2$. Find the separation in time between when the bulbs flashed as seen by the train passenger seated in the middle of the car.

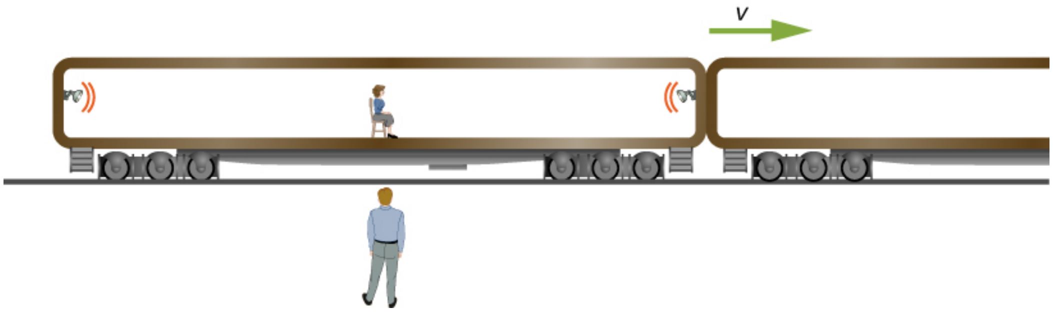


Figure 5.14 An person watching a train go by observes two bulbs flash simultaneously at opposite ends of a passenger car. There is another passenger inside of the car observing the same flashes but from a different perspective.

Solution

a. Identify the known: $\Delta t = 0$.

   Note that the spatial separation of the two events is between the two lamps, not the distance of the lamp to the passenger.

b. Identify the unknown: $\Delta t' = t'_2 - t'_1$.

   Again, note that the time interval is between the flashes of the lamps, not between arrival times for reaching the passenger.

c. Express the answer as an equation:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}.$$

d. Do the calculation:

$$\begin{aligned} 0 &= \frac{\Delta t' + \frac{c}{2}(26\text{ m})/c^2}{\sqrt{1 - v^2/c^2}} \\ \Delta t' &= -\frac{26\text{ m/s}}{2c} = -\frac{26\text{ m/s}}{2\left(3.00\times10^8\text{ m/s}\right)} \\ \Delta t' &= -4.33\times10^{-8}\text{ s}. \end{aligned}$$

Significance

The sign indicates that the event with the larger x_2', namely, the flash from the right, is seen to occur first in the S' frame, as found earlier for this example, so that $t_2 < t_1$.

Space-time

Relativistic phenomena can be analyzed in terms of events in a four-dimensional space-time. When phenomena such as the twin paradox, time dilation, length contraction, and the dependence of simultaneity on relative motion are viewed in this way, they are seen to be characteristic of the nature of space and time, rather than specific aspects of electromagnetism.

In three-dimensional space, positions are specified by three coordinates on a set of Cartesian axes, and the displacement of one point from another is given by:

$$(\Delta x,\ \Delta y,\ \Delta z) = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1).$$

The distance Δr between the points is

$$\Delta r^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

The distance Δr is invariant under a rotation of axes. If a new set of Cartesian axes rotated around the origin relative to the original axes are used, each point in space will have new coordinates in terms of the new axes, but the distance $\Delta r'$ given by

$$\Delta r'^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2.$$

That has the same value that Δr^2 had. Something similar happens with the Lorentz transformation in space-time.

Define the separation between two events, each given by a set of *x*, *y*, *z*¸ and *ct* along a four-dimensional Cartesian system of axes in space-time, as

$$(\Delta x,\ \Delta y,\ \Delta z,\ c\Delta t) = \left(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1,\ c(t_2 - t_1)\right).$$

Also define the space-time interval Δs between the two events as

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

If the two events have the same value of *ct* in the frame of reference considered, Δs would correspond to the distance Δr between points in space.

The path of a particle through space-time consists of the events (*x*, *y*, *z*¸ *ct*) specifying a location at each time of its motion. The path through space-time is called the **world line** of the particle. The world line of a particle that remains at rest at the same location is a straight line that is parallel to the time axis. If the particle moves at constant velocity parallel to the *x*-axis, its world line would be a sloped line $x = vt$, corresponding to a simple displacement vs. time graph. If the particle accelerates, its world line is curved. The increment of *s* along the world line of the particle is given in differential form as

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2.$$

Just as the distance Δr is invariant under rotation of the space axes, the space-time interval:

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

is invariant under the Lorentz transformation. This follows from the postulates of relativity, and can be seen also by substitution of the previous Lorentz transformation equations into the expression for the space-time interval:

$$\begin{aligned}
\Delta s^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2 \\
&= \left(\frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}\right)^2 + (\Delta y')^2 + (\Delta z')^2 - \left(c\frac{\Delta t' + \frac{v\Delta x'}{c^2}}{\sqrt{1 - v^2/c^2}}\right)^2 \\
&= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (c\Delta t')^2 \\
&= \Delta s'^2.
\end{aligned}$$

In addition, the Lorentz transformation changes the coordinates of an event in time and space similarly to how a three-dimensional rotation changes old coordinates into new coordinates:

Lorentz transformation	**Axis – rotation around *z*-axis**
$(x,\ t$ coordinates):	$(x,\ y$ coordinates):
$x' = (\gamma)x + (-\beta\gamma)ct$	$x' = (\cos\theta)x + (\sin\theta)y$
$ct' = (-\beta\gamma)x + (\gamma)ct$	$y' = (-\sin\theta)x + (\cos\theta)y$

where $\gamma = \dfrac{1}{\sqrt{1 - \beta^2}};\quad \beta = v/c.$

Lorentz transformations can be regarded as generalizations of spatial rotations to space-time. However, there are some differences between a three-dimensional axis rotation and a Lorentz transformation involving the time axis, because of differences in how the metric, or rule for measuring the displacements Δr and Δs, differ. Although Δr is invariant under spatial rotations and Δs is invariant also under Lorentz transformation, the Lorentz transformation involving the time axis does not preserve some features, such as the axes remaining perpendicular or the length scale along each axis remaining the same.

Note that the quantity Δs^2 can have either sign, depending on the coordinates of the space-time events involved. For pairs of events that give it a negative sign, it is useful to define $\Delta\tau^2$ as $-\Delta s^2$. The significance of $\Delta\tau$ as just defined follows by noting that in a frame of reference where the two events occur at the same location, we have $\Delta x = \Delta y = \Delta z = 0$ and therefore (from the equation for $\Delta s^2 = -\Delta\tau^2$):

$$\Delta\tau^2 = -\Delta s^2 = (\Delta t)^2.$$

Therefore $\Delta\tau$ is the time interval Δt in the frame of reference where both events occur at the same location. It is the same interval of proper time discussed earlier. It also follows from the relation between Δs and that $\Delta\tau$ that because Δs is Lorentz invariant, the proper time is also Lorentz invariant. All observers in all inertial frames agree on the proper time intervals between the same two events.

5.5 Check Your Understanding Show that if a time increment *dt* elapses for an observer who sees the particle moving with velocity *v*, it corresponds to a proper time particle increment for the particle of $d\tau = \gamma dt$.

The light cone

We can deal with the difficulty of visualizing and sketching graphs in four dimensions by imagining the three spatial coordinates to be represented collectively by a horizontal axis, and the vertical axis to be the *ct*-axis. Starting with a particular event in space-time as the origin of the space-time graph shown, the world line of a particle that remains at rest at the initial location of the event at the origin then is the time axis. Any plane through the time axis parallel to the spatial axes contains all the events that are simultaneous with each other and with the intersection of the plane and the time axis, as seen in the rest frame of the event at the origin.

It is useful to picture a light cone on the graph, formed by the world lines of all light beams passing through the origin event *A*, as shown in **Figure 5.15**. The light cone, according to the postulates of relativity, has sides at an angle of $45°$ if the time axis is measured in units of *ct*, and, according to the postulates of relativity, the light cone remains the same in all inertial frames. Because the event *A* is arbitrary, every point in the space-time diagram has a light cone associated with it.

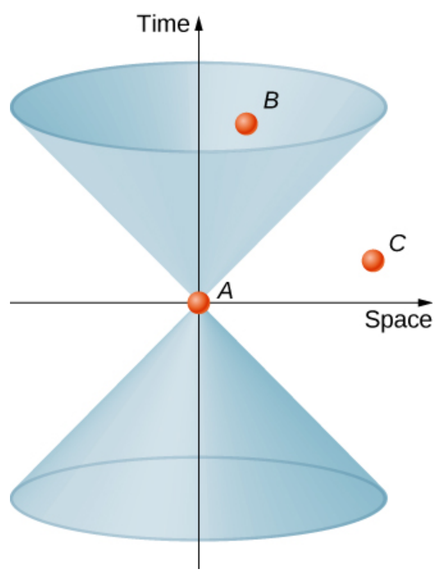


Figure 5.15 The light cone consists of all the world lines followed by light from the event *A* at the vertex of the cone.

Consider now the world line of a particle through space-time. Any world line outside of the cone, such as one passing from *A* through *C*, would involve speeds greater than *c*, and would therefore not be possible. Events such as *C* that lie outside the light cone are said to have a space-like separation from event A. They are characterized by:

$$\Delta s_{AC}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 > 0.$$

An event like B that lies in the upper cone is reachable without exceeding the speed of light in vacuum, and is characterized by

$$\Delta s_{AB}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 < 0.$$

The event is said to have a time-like separation from A. Time-like events that fall into the upper half of the light cone occur at greater values of t than the time of the event A at the vertex and are in the future relative to A. Events that have time-like separation from A and fall in the lower half of the light cone are in the past, and can affect the event at the origin. The region outside the light cone is labeled as neither past nor future, but rather as "elsewhere."

For any event that has a space-like separation from the event at the origin, it is possible to choose a time axis that will make the two events occur at the same time, so that the two events are simultaneous in some frame of reference. Therefore, which of the events with space-like separation comes before the other in time also depends on the frame of reference of the observer. Since space-like separations can be traversed only by exceeding the speed of light; this violation of which event can cause the other provides another argument for why particles cannot travel faster than the speed of light, as well as potential material for science fiction about time travel. Similarly for any event with time-like separation from the event at the origin, a frame of reference can be found that will make the events occur at the same location. Because the relations

$$\Delta s_{AC}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 > 0$$

and

$$\Delta s_{AB}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 < 0.$$

are Lorentz invariant, whether two events are time-like and can be made to occur at the same place or space-like and can be made to occur at the same time is the same for all observers. All observers in different inertial frames of reference agree on whether two events have a time-like or space-like separation.

The twin paradox seen in space-time

The twin paradox discussed earlier involves an astronaut twin traveling at near light speed to a distant star system, and returning to Earth. Because of time dilation, the space twin is predicted to age much less than the earthbound twin. This seems paradoxical because we might have expected at first glance for the relative motion to be symmetrical and naively thought it possible to also argue that the earthbound twin should age less.

To analyze this in terms of a space-time diagram, assume that the origin of the axes used is fixed in Earth. The world line of the earthbound twin is then along the time axis.

The world line of the astronaut twin, who travels to the distant star and then returns, must deviate from a straight line path in order to allow a return trip. As seen in **Figure 5.16**, the circumstances of the two twins are not at all symmetrical. Their paths in space-time are of manifestly different length. Specifically, the world line of the earthbound twin has length $2c\Delta t$, which then gives the proper time that elapses for the earthbound twin as $2\Delta t$. The distance to the distant star system is $\Delta x = v\Delta t$. The proper time that elapses for the space twin is $2\Delta\tau$ where

$$c^2\Delta\tau^2 = -\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2.$$

This is considerably shorter than the proper time for the earthbound twin by the ratio

$$\begin{aligned}\frac{c\Delta\tau}{c\Delta t} &= \sqrt{\frac{(c\Delta t)^2 - (\Delta x)^2}{(c\Delta t)^2}} = \sqrt{\frac{(c\Delta t)^2 - (v\Delta t)^2}{(c\Delta t)^2}} \\ &= \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}.\end{aligned}$$

consistent with the time dilation formula. The twin paradox is therefore seen to be no paradox at all. The situation of the two twins is not symmetrical in the space-time diagram. The only surprise is perhaps that the seemingly longer path on the space-time diagram corresponds to the smaller proper time interval, because of how $\Delta\tau$ and Δs depend on Δx and Δt.

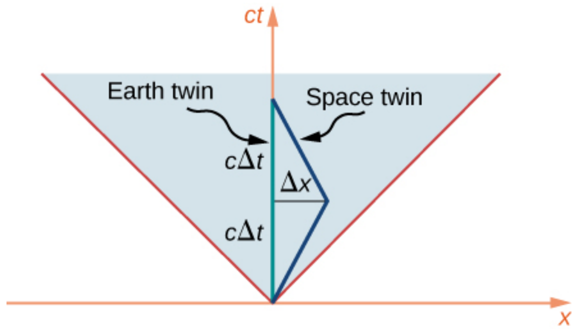


Figure 5.16 The space twin and the earthbound twin, in the twin paradox example, follow world lines of different length through space-time.

Lorentz transformations in space-time

We have already noted how the Lorentz transformation leaves

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

unchanged and corresponds to a rotation of axes in the four-dimensional space-time. If the S and S' frames are in relative motion along their shared *x*-direction the space and time axes of S' are rotated by an angle α as seen from S, in the way shown in shown in **Figure 5.17**, where:

$$\tan\alpha = \frac{v}{c} = \beta.$$

This differs from a rotation in the usual three-dimension sense, insofar as the two space-time axes rotate toward each other symmetrically in a scissors-like way, as shown. The rotation of the time and space axes are both through the same angle. The mesh of dashed lines parallel to the two axes show how coordinates of an event would be read along the primed axes. This would be done by following a line parallel to the x' and one parallel to the t'-axis, as shown by the dashed lines. The length scale of both axes are changed by:

$$ct' = ct\sqrt{\frac{1+\beta^2}{1-\beta^2}}; \qquad x' = x\sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

The line labeled "$v = c$" at $45°$ to the *x*-axis corresponds to the edge of the light cone, and is unaffected by the Lorentz transformation, in accordance with the second postulate of relativity. The "$v = c$" line, and the light cone it represents, are the same for both the *S* and S' frame of reference.

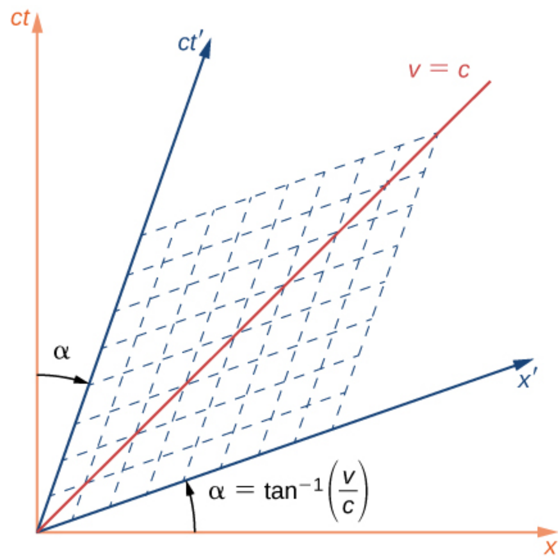


Figure 5.17 The Lorentz transformation results in new space and time axes rotated in a scissors-like way with respect to the original axes.

Simultaneity

Simultaneity of events at separated locations depends on the frame of reference used to describe them, as given by the scissors-like "rotation" to new time and space coordinates as described. If two events have the same t values in the unprimed frame of reference, they need not have the same values measured along the ct'-axis, and would then not be simultaneous in the primed frame.

As a specific example, consider the near-light-speed train in which flash lamps at the two ends of the car have flashed simultaneously in the frame of reference of an observer on the ground. The space-time graph is shown **Figure 5.18**. The flashes of the two lamps are represented by the dots labeled "Left flash lamp" and "Right flash lamp" that lie on the light cone in the past. The world line of both pulses travel along the edge of the light cone to arrive at the observer on the ground simultaneously. Their arrival is the event at the origin. They therefore had to be emitted simultaneously in the unprimed frame, as represented by the point labeled as t(both). But time is measured along the ct'-axis in the frame of reference of the observer seated in the middle of the train car. So in her frame of reference, the emission event of the bulbs labeled as t' (left) and t' (right) were not simultaneous.

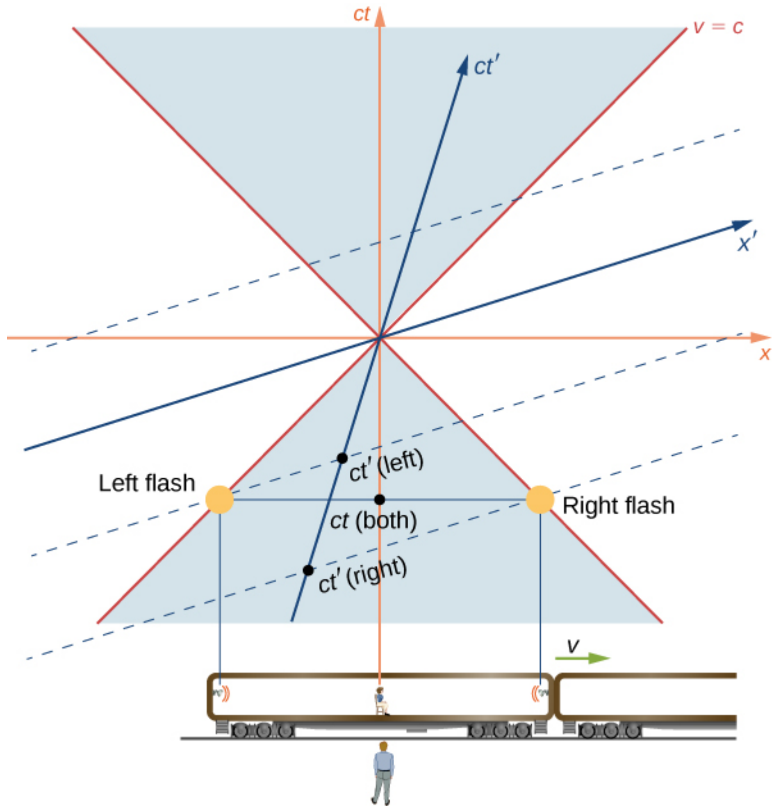


Figure 5.18 The train example revisited. The flashes occur at the same time t(both) along the time axis of the ground observer, but at different times, along the t' time axis of the passenger.

In terms of the space-time diagram, the two observers are merely using different time axes for the same events because they are in different inertial frames, and the conclusions of both observers are equally valid. As the analysis in terms of the space-time diagrams further suggests, the property of how simultaneity of events depends on the frame of reference results from the properties of space and time itself, rather than from anything specifically about electromagnetism.

5.6 | Relativistic Velocity Transformation

Learning Objectives

By the end of this section, you will be able to:

- Derive the equations consistent with special relativity for transforming velocities in one inertial frame of reference into another.
- Apply the velocity transformation equations to objects moving at relativistic speeds.
- Examine how the combined velocities predicted by the relativistic transformation equations compare with those expected classically.

Remaining in place in a kayak in a fast-moving river takes effort. The river current pulls the kayak along. Trying to paddle against the flow can move the kayak upstream relative to the water, but that only accounts for part of its velocity relative to the shore. The kayak's motion is an example of how velocities in Newtonian mechanics combine by vector addition. The kayak's velocity is the vector sum of its velocity relative to the water and the water's velocity relative to the riverbank. However, the relativistic addition of velocities is quite different.

Velocity Transformations

Imagine a car traveling at night along a straight road, as in **Figure 5.19**. The driver sees the light leaving the headlights at speed c within the car's frame of reference. If the Galilean transformation applied to light, then the light from the car's headlights would approach the pedestrian at a speed $u = v + c,$ contrary to Einstein's postulates.

Figure 5.19 According to experimental results and the second postulate of relativity, light from the car's headlights moves away from the car at speed *c* and toward the observer on the sidewalk at speed *c*.

Both the distance traveled and the time of travel are different in the two frames of reference, and they must differ in a way that makes the speed of light the same in all inertial frames. The correct rules for transforming velocities from one frame to another can be obtained from the Lorentz transformation equations.

Relativistic Transformation of Velocity

Suppose an object *P* is moving at constant velocity $\mathbf{u} = \left(u'_x,\, u'_y,\, u'_z\right)$ as measured in the S' frame. The S' frame is moving along its x'-axis at velocity *v*. In an increment of time dt', the particle is displaced by dx' along the x'-axis. Applying the Lorentz transformation equations gives the corresponding increments of time and displacement in the unprimed axes:

$$\begin{aligned} dt &= \gamma\left(dt' + v dx'/c^2\right) \\ dx &= \gamma(dx' + v dt') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

The velocity components of the particle seen in the unprimed coordinate system are then

$$\begin{aligned} \frac{dx}{dt} &= \frac{\gamma(dx' + v dt')}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \\ \frac{dy}{dt} &= \frac{dy'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \\ \frac{dz}{dt} &= \frac{dz'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)}. \end{aligned}$$

We thus obtain the equations for the velocity components of the object as seen in frame *S*:

$$u_x = \left(\frac{u'_x + v}{1 + v u'_x/c^2}\right), \quad u_y = \left(\frac{u'_y/\gamma}{1 + v u'_x/c^2}\right), \quad u_z = \left(\frac{u'_z/\gamma}{1 + v u'_x/c^2}\right).$$

Compare this with how the Galilean transformation of classical mechanics says the velocities transform, by adding simply as vectors:

$$u_x = u'_x + u, \quad u_y = u'_y, \quad u_z = u'_z.$$

When the relative velocity of the frames is much smaller than the speed of light, that is, when $v \ll c,$ the special relativity velocity addition law reduces to the Galilean velocity law. When the speed v of S' relative to S is comparable to the speed of light, the **relativistic velocity addition** law gives a much smaller result than the **classical (Galilean) velocity addition** does.

Example 5.9

Velocity Transformation Equations for Light

Suppose a spaceship heading directly toward Earth at half the speed of light sends a signal to us on a laser-produced beam of light (Figure 5.20). Given that the light leaves the ship at speed c as observed from the ship, calculate the speed at which it approaches Earth.

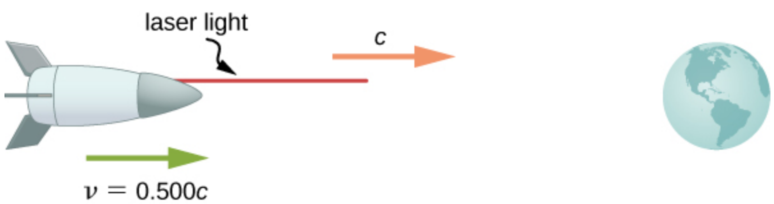


Figure 5.20 How fast does a light signal approach Earth if sent from a spaceship traveling at $0.500c$?

Strategy

Because the light and the spaceship are moving at relativistic speeds, we cannot use simple velocity addition. Instead, we determine the speed at which the light approaches Earth using relativistic velocity addition.

Solution

a. Identify the knowns: $v = 0.500c$; $u' = c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + c}{1 + \frac{(0.500c)(c)}{c^2}} \\ &= \frac{(0.500 + 1)c}{\left(\frac{c^2 + 0.500c^2}{c^2}\right)} \\ &= c. \end{aligned}$$

Significance

Relativistic velocity addition gives the correct result. Light leaves the ship at speed c and approaches Earth at speed c. The speed of light is independent of the relative motion of source and observer, whether the observer is on the ship or earthbound.

Velocities cannot add to greater than the speed of light, provided that v is less than c and u' does not exceed c. The following example illustrates that relativistic velocity addition is not as symmetric as classical velocity addition.

Example 5.10

Relativistic Package Delivery

Suppose the spaceship in the previous example approaches Earth at half the speed of light and shoots a canister at a speed of $0.750c$ (Figure 5.21). (a) At what velocity does an earthbound observer see the canister if it is shot directly toward Earth? (b) If it is shot directly away from Earth?

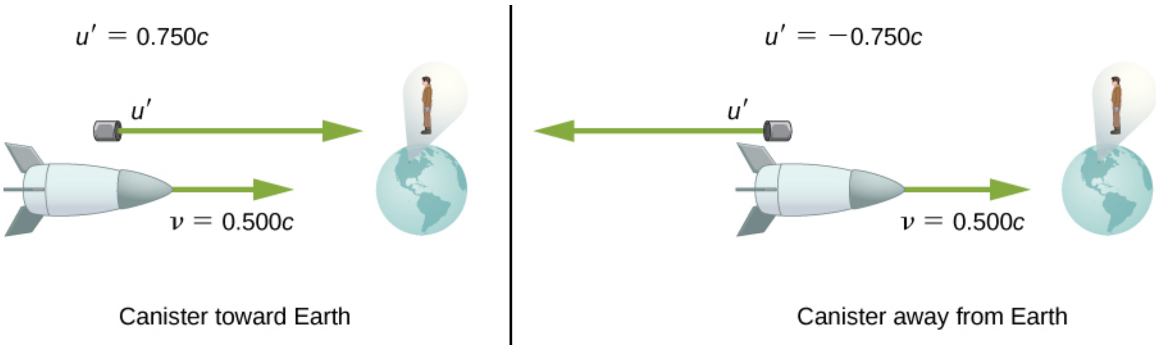


Figure 5.21 A canister is fired at $0.7500c$ toward Earth or away from Earth.

Strategy

Because the canister and the spaceship are moving at relativistic speeds, we must determine the speed of the canister by an earthbound observer using relativistic velocity addition instead of simple velocity addition.

Solution for (a)

a. Identify the knowns: $v = 0.500c;\ u' = 0.750c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + 0.750c}{1 + \frac{(0.500c)(0.750c)}{c^2}} \\ &= 0.909c. \end{aligned}$$

Solution for (b)

a. Identify the knowns: $v = 0.500c;\ u' = -0.750c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + (-0.750c)}{1 + \frac{(0.500c)(-0.750c)}{c^2}} \\ &= -0.400c. \end{aligned}$$

Significance

The minus sign indicates a velocity away from Earth (in the opposite direction from v), which means the canister is heading toward Earth in part (a) and away in part (b), as expected. But relativistic velocities do not add as simply as they do classically. In part (a), the canister does approach Earth faster, but at less than the vector sum of the velocities, which would give $1.250c$. In part (b), the canister moves away from Earth at a velocity of $-0.400c$, which is *faster* than the $-0.250c$ expected classically. The differences in velocities are not even symmetric: In part (a), an observer on Earth sees the canister and the ship moving apart at a speed of $0.409c$, and at a speed of $0.900c$ in part (b).

5.6 Check Your Understanding Distances along a direction perpendicular to the relative motion of the two frames are the same in both frames. Why then are velocities perpendicular to the x-direction different in the two frames?

5.7 | Doppler Effect for Light

Learning Objectives

By the end of this section, you will be able to:

- Explain the origin of the shift in frequency and wavelength of the observed wavelength when observer and source moved toward or away from each other
- Derive an expression for the relativistic Doppler shift
- Apply the Doppler shift equations to real-world examples

As discussed in the chapter on sound, if a source of sound and a listener are moving farther apart, the listener encounters fewer cycles of a wave in each second, and therefore lower frequency, than if their separation remains constant. For the same reason, the listener detects a higher frequency if the source and listener are getting closer. The resulting Doppler shift in detected frequency occurs for any form of wave. For sound waves, however, the equations for the Doppler shift differ markedly depending on whether it is the source, the observer, or the air, which is moving. Light requires no medium, and the Doppler shift for light traveling in vacuum depends only on the relative speed of the observer and source.

The Relativistic Doppler Effect

Suppose an observer in S sees light from a source in S' moving away at velocity v (**Figure 5.22**). The wavelength of the light could be measured within S'—for example, by using a mirror to set up standing waves and measuring the distance between nodes. These distances are proper lengths with S' as their rest frame, and change by a factor $\sqrt{1 - v^2/c^2}$ when measured in the observer's frame S, where the ruler measuring the wavelength in S' is seen as moving.

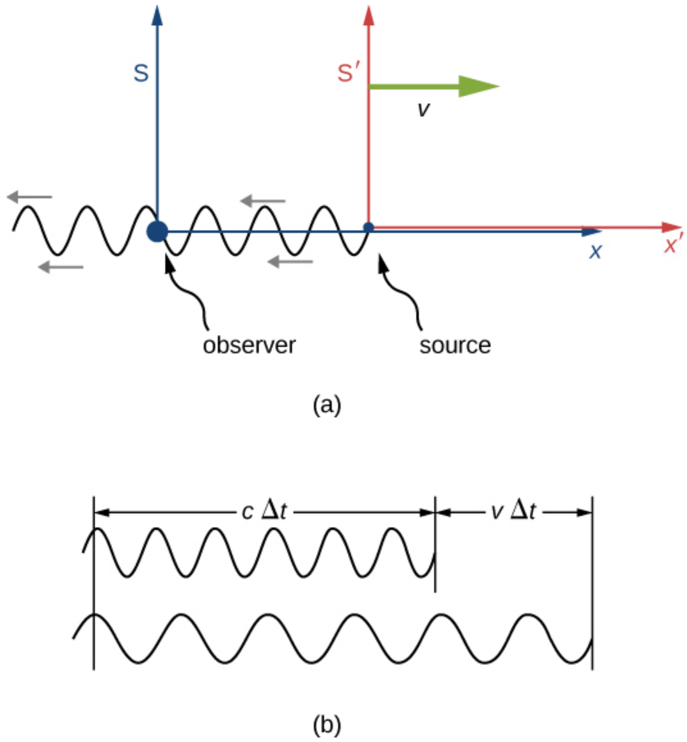


Figure 5.22 (a) When a light wave is emitted by a source fixed in the moving inertial frame S', the observer in S sees the wavelength measured in S'. to be shorter by a factor $\sqrt{1 - v^2/c^2}$. (b) Because the observer sees the source moving away within S, the wave pattern reaching the observer in S is also stretched by the factor $(c\Delta t + v\Delta t)/(c\Delta t) = 1 + v/c$.

If the source were stationary in S, the observer would see a length $c\Delta t$ of the wave pattern in time Δt. But because of the motion of S' relative to S, considered solely within S, the observer sees the wave pattern, and therefore the wavelength, stretched out by a factor of

$$\frac{c\Delta t_{\text{period}} + v\Delta t_{\text{period}}}{c\Delta t_{\text{period}}} = 1 + \frac{v}{c}$$

as illustrated in (b) of **Figure 5.22**. The overall increase from both effects gives

$$\lambda_{\text{obs}} = \lambda_{\text{src}}\left(1 + \frac{v}{c}\right)\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \lambda_{\text{src}}\left(1 + \frac{v}{c}\right)\sqrt{\frac{1}{\left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)}} = \lambda_{\text{src}}\sqrt{\frac{\left(1 + \frac{v}{c}\right)}{\left(1 - \frac{v}{c}\right)}}$$

where λ_{src} is the wavelength of the light seen by the source in S' and λ_{obs} is the wavelength that the observer detects within S.

Red Shifts and Blue Shifts

The observed wavelength λ_{obs} of electromagnetic radiation is longer (called a "red shift") than that emitted by the source when the source moves away from the observer. Similarly, the wavelength is shorter (called a "blue shift") when the source moves toward the observer. The amount of change is determined by

$$\lambda_{\text{obs}} = \lambda_s\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

where λ_s is the wavelength in the frame of reference of the source, and v is the relative velocity of the two frames S and

S'. The velocity v is positive for motion away from an observer and negative for motion toward an observer. In terms of source frequency and observed frequency, this equation can be written as

$$f_{\text{obs}} = f_s \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}.$$

Notice that the signs are different from those of the wavelength equation.

Example 5.11

Calculating a Doppler Shift

Suppose a galaxy is moving away from Earth at a speed $0.825c$. It emits radio waves with a wavelength of 0.525 m. What wavelength would we detect on Earth?

Strategy

Because the galaxy is moving at a relativistic speed, we must determine the Doppler shift of the radio waves using the relativistic Doppler shift instead of the classical Doppler shift.

Solution

a. Identify the knowns: $u = 0.825c$; $\lambda_s = 0.525 \text{ m}$.

b. Identify the unknown: λ_{obs}.

c. Express the answer as an equation:

$$\lambda_{\text{obs}} = \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

d. Do the calculation:

$$\begin{aligned} \lambda_{\text{obs}} &= \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \\ &= (0.525 \text{ m})\sqrt{\frac{1 + \frac{0.825c}{c}}{1 - \frac{0.825c}{c}}} \\ &= 1.70 \text{ m}. \end{aligned}$$

Significance

Because the galaxy is moving away from Earth, we expect the wavelengths of radiation it emits to be redshifted. The wavelength we calculated is 1.70 m, which is redshifted from the original wavelength of 0.525 m. You will see in **Particle Physics and Cosmology** that detecting redshifted radiation led to present-day understanding of the origin and evolution of the universe.

5.7 Check Your Understanding Suppose a space probe moves away from Earth at a speed $0.350c$. It sends a radio-wave message back to Earth at a frequency of 1.50 GHz. At what frequency is the message received on Earth?

The relativistic Doppler effect has applications ranging from Doppler radar storm monitoring to providing information on the motion and distance of stars. We describe some of these applications in the exercises.

5.8 | Relativistic Momentum

Learning Objectives

By the end of this section, you will be able to:

- Define relativistic momentum in terms of mass and velocity
- Show how relativistic momentum relates to classical momentum
- Show how conservation of relativistic momentum limits objects with mass to speeds less than c

Momentum is a central concept in physics. The broadest form of Newton's second law is stated in terms of momentum. Momentum is conserved whenever the net external force on a system is zero. This makes momentum conservation a fundamental tool for analyzing collisions (**Figure 5.23**). Much of what we know about subatomic structure comes from the analysis of collisions of accelerator-produced relativistic particles, and momentum conservation plays a crucial role in this analysis.

Figure 5.23 Momentum is an important concept for these football players from the University of California at Berkeley and the University of California at Davis. A player with the same velocity but greater mass collides with greater impact because his momentum is greater. For objects moving at relativistic speeds, the effect is even greater.

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Does the law of conservation of momentum survive this requirement at high velocities? It can be shown that the momentum calculated as merely $\vec{\mathbf{p}} = m\frac{d\vec{\mathbf{x}}}{dt}$, even if it is conserved in one frame of reference, may not be conserved in another after applying the Lorentz transformation to the velocities. The correct equation for momentum can be shown, instead, to be the classical expression in terms of the increment $d\tau$ of proper time of the particle, observed in the particle's rest frame:

$$\begin{aligned}\vec{\mathbf{p}} &= m\frac{d\vec{\mathbf{x}}}{d\tau} = m\frac{d\vec{\mathbf{x}}}{dt}\frac{dt}{d\tau} \\ &= m\frac{d\vec{\mathbf{x}}}{dt}\frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{m\vec{\mathbf{u}}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{\mathbf{u}}\,.\end{aligned}$$

Relativistic Momentum

Relativistic momentum $\vec{\mathbf{p}}$ is classical momentum multiplied by the relativistic factor γ:

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{u}} \tag{5.6}$$

where m is the **rest mass** of the object, $\vec{\mathbf{u}}$ is its velocity relative to an observer, and γ is the relativistic factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{5.7}$$

Note that we use u for velocity here to distinguish it from relative velocity v between observers. The factor γ that occurs here has the same form as the previous relativistic factor γ except that it is now in terms of the velocity of the particle u instead of the relative velocity v of two frames of reference.

With p expressed in this way, total momentum p_{tot} is conserved whenever the net external force is zero, just as in classical physics. Again we see that the relativistic quantity becomes virtually the same as the classical quantity at low velocities, where u/c is small and γ is very nearly equal to 1. Relativistic momentum has the same intuitive role as classical momentum. It is greatest for large masses moving at high velocities, but because of the factor γ, relativistic momentum approaches infinity as u approaches c (**Figure 5.24**). This is another indication that an object with mass cannot reach the speed of light. If it did, its momentum would become infinite—an unreasonable value.

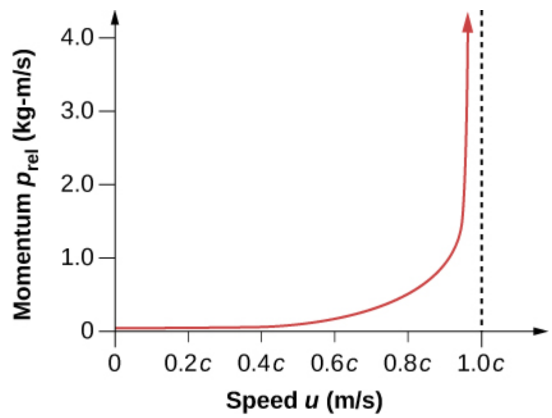


Figure 5.24 Relativistic momentum approaches infinity as the velocity of an object approaches the speed of light.

The relativistically correct definition of momentum as $p = \gamma mu$ is sometimes taken to imply that mass varies with velocity: $m_{\text{var}} = \gamma m,$ particularly in older textbooks. However, note that m is the mass of the object as measured by a person at rest relative to the object. Thus, m is defined to be the rest mass, which could be measured at rest, perhaps using gravity. When a mass is moving relative to an observer, the only way that its mass can be determined is through collisions or other means involving momentum. Because the mass of a moving object cannot be determined independently of momentum, the only meaningful mass is rest mass. Therefore, when we use the term "mass," assume it to be identical to "rest mass."

Relativistic momentum is defined in such a way that conservation of momentum holds in all inertial frames. Whenever the net external force on a system is zero, relativistic momentum is conserved, just as is the case for classical momentum. This has been verified in numerous experiments.

5.8 Check Your Understanding What is the momentum of an electron traveling at a speed 0.985*c*? The rest mass of the electron is 9.11×10^{-31} kg.

5.9 | Relativistic Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain how the work-energy theorem leads to an expression for the relativistic kinetic energy of an object
- Show how the relativistic energy relates to the classical kinetic energy, and sets a limit on the speed of any object with mass
- Describe how the total energy of a particle is related to its mass and velocity
- Explain how relativity relates to energy-mass equivalence, and some of the practical implications of energy-mass equivalence

The tokamak in **Figure 5.25** is a form of experimental fusion reactor, which can change mass to energy. Nuclear reactors are proof of the relationship between energy and matter.

Conservation of energy is one of the most important laws in physics. Not only does energy have many important forms, but each form can be converted to any other. We know that classically, the total amount of energy in a system remains constant. Relativistically, energy is still conserved, but energy-mass equivalence must now be taken into account, for example, in the reactions that occur within a nuclear reactor. Relativistic energy is intentionally defined so that it is conserved in all inertial frames, just as is the case for relativistic momentum. As a consequence, several fundamental quantities are related in ways not known in classical physics. All of these relationships have been verified by experimental results and have fundamental consequences. The altered definition of energy contains some of the most fundamental and spectacular new insights into nature in recent history.

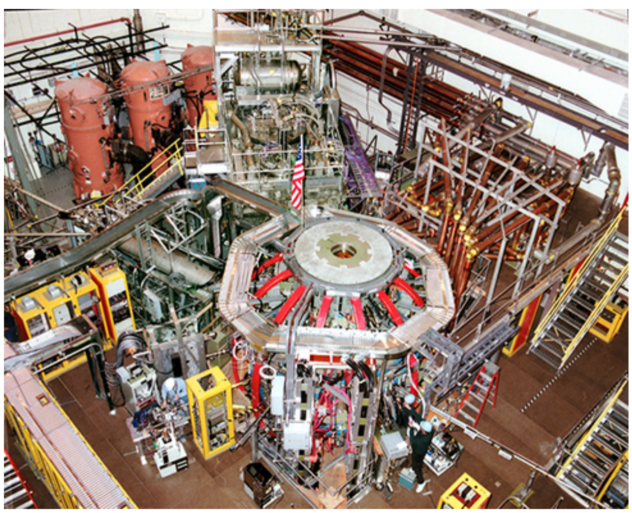


Figure 5.25 The National Spherical Torus Experiment (NSTX) is a fusion reactor in which hydrogen isotopes undergo fusion to produce helium. In this process, a relatively small mass of fuel is converted into a large amount of energy. (credit: Princeton Plasma Physics Laboratory)

Kinetic Energy and the Ultimate Speed Limit

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Einstein showed that the law of conservation of energy of a particle is valid relativistically, but for energy expressed in terms of velocity and mass in a way consistent with relativity.

Consider first the relativistic expression for the kinetic energy. We again use u for velocity to distinguish it from relative velocity v between observers. Classically, kinetic energy is related to mass and speed by the familiar expression $K = \frac{1}{2}mu^2$. The corresponding relativistic expression for kinetic energy can be obtained from the work-energy theorem. This theorem states that the net work on a system goes into kinetic energy. Specifically, if a force, expressed as $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = m\frac{d(\gamma \vec{\mathbf{u}})}{dt}$, accelerates a particle from rest to its final velocity, the work done on the particle should be equal to its final kinetic energy. In mathematical form, for one-dimensional motion:

$$\begin{aligned} K &= \int F dx = \int m\frac{d}{dt}(\gamma u)dx \\ &= m\int \frac{d(\gamma u)}{dt}\frac{dx}{dt}dt = m\int u\frac{d}{dt}\left(\frac{u}{\sqrt{1-(u/c)^2}}\right)dt. \end{aligned}$$

Integrate this by parts to obtain

$$\begin{aligned} K &= \frac{mu^2}{\sqrt{1-(u/c)^2}}\Bigg|_{0u} - m\int \frac{u}{\sqrt{1-(u/c)^2}}\frac{du}{dt}dt \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - m\int \frac{u}{\sqrt{1-(u/c)^2}}du \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2\left(\sqrt{1-(u/c)^2}\right)\Bigg|_0^u \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} + \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2 \\ &= mc^2\left[\frac{(u^2/c^2)+1-(u^2/c^2)}{\sqrt{1-(u/c)^2}}\right] - mc^2 \\ K &= \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2. \end{aligned}$$

Relativistic Kinetic Energy

Relativistic kinetic energy of any particle of mass m is

$$K_{\text{rel}} = (\gamma - 1)mc^2. \quad (5.8)$$

When an object is motionless, its speed is $u = 0$ and

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1$$

so that $K_{\text{rel}} = 0$ at rest, as expected. But the expression for relativistic kinetic energy (such as total energy and rest energy) does not look much like the classical $\frac{1}{2}mu^2$. To show that the expression for K_{rel} reduces to the classical expression for kinetic energy at low speeds, we use the binomial expansion to obtain an approximation for $(1+\varepsilon)^n$ valid for small ε:

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + \frac{n(n-1)(n-2)}{3!}\varepsilon^3 + \cdots \approx 1 + n\varepsilon$$

by neglecting the very small terms in ε^2 and higher powers of ε. Choosing $\varepsilon = -u^2/c^2$ and $n = -\frac{1}{2}$ leads to the conclusion that γ at nonrelativistic speeds, where $\varepsilon = u/c$ is small, satisfies

$$\gamma = \left(1 - u^2/c^2\right)^{-1/2} \approx 1 + \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

A binomial expansion is a way of expressing an algebraic quantity as a sum of an infinite series of terms. In some cases, as in the limit of small speed here, most terms are very small. Thus, the expression derived here for γ is not exact, but it is a very accurate approximation. Therefore, at low speed:

$$\gamma - 1 = \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

Entering this into the expression for relativistic kinetic energy gives

$$K_{\text{rel}} = \left[\frac{1}{2}\left(\frac{u^2}{c^2}\right)\right]mc^2 = \frac{1}{2}mu^2 = K_{\text{class}}.$$

That is, relativistic kinetic energy becomes the same as classical kinetic energy when $u << c$.

It is even more interesting to investigate what happens to kinetic energy when the speed of an object approaches the speed of light. We know that γ becomes infinite as *u* approaches *c*, so that K_{rel} also becomes infinite as the velocity approaches the speed of light (**Figure 5.26**). The increase in K_{rel} is far larger than in K_{class} as *v* approaches *c*. An infinite amount of work (and, hence, an infinite amount of energy input) is required to accelerate a mass to the speed of light.

The Speed of Light

No object with mass can attain the **speed of light**.

The speed of light is the ultimate speed limit for any particle having mass. All of this is consistent with the fact that velocities less than *c* always add to less than *c*. Both the relativistic form for kinetic energy and the ultimate speed limit being *c* have been confirmed in detail in numerous experiments. No matter how much energy is put into accelerating a mass, its velocity can only approach—not reach—the speed of light.

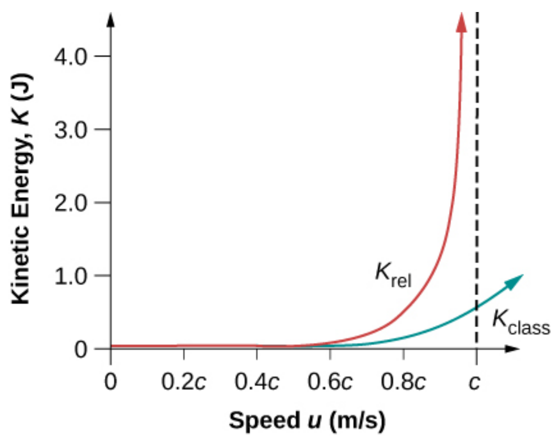


Figure 5.26 This graph of K_{rel} versus velocity shows how kinetic energy increases without bound as velocity approaches the speed of light. Also shown is K_{class}, the classical kinetic energy.

Example 5.12

Comparing Kinetic Energy

An electron has a velocity $v = 0.990c$. (a) Calculate the kinetic energy in MeV of the electron. (b) Compare this with the classical value for kinetic energy at this velocity. (The mass of an electron is $9.11 \times 10^{-31}\,\text{kg}$.)

Strategy

The expression for relativistic kinetic energy is always correct, but for (a), it must be used because the velocity is highly relativistic (close to c). First, we calculate the relativistic factor γ, and then use it to determine the relativistic kinetic energy. For (b), we calculate the classical kinetic energy (which would be close to the relativistic value if v were less than a few percent of c) and see that it is not the same.

Solution for (a)

For part (a):

a. Identify the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31}\,\text{kg}$.

b. Identify the unknown: K_{rel}.

c. Express the answer as an equation: $K_{\text{rel}} = (\gamma - 1)mc^2$ with $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$.

d. Do the calculation. First calculate γ. Keep extra digits because this is an intermediate calculation:

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\\ &= \frac{1}{\sqrt{1 - \frac{(0.990c)^2}{c^2}}}\\ &= 7.0888.\end{aligned}$$

Now use this value to calculate the kinetic energy:

$$\begin{aligned}K_{\text{rel}} &= (\gamma - 1)mc^2\\ &= (7.0888 - 1)(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^8\ \text{m/s}^2)\\ &= 4.9922 \times 10^{-13}\ \text{J}.\end{aligned}$$

e. Convert units:

$$\begin{aligned}K_{\text{rel}} &= (4.9922 \times 10^{-13}\ \text{J})\left(\frac{1\ \text{MeV}}{1.60 \times 10^{-13}\ \text{J}}\right)\\ &= 3.12\ \text{MeV}.\end{aligned}$$

Solution for (b)

For part (b):

a. List the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31}\,\text{kg}$.

b. List the unknown: K_{rel}.

c. Express the answer as an equation: $K_{\text{class}} = \frac{1}{2}mu^2$.

d. Do the calculation:

$$\begin{aligned}K_{\text{class}} &= \frac{1}{2}mu^2\\ &= \frac{1}{2}(9.11 \times 10^{-31}\ \text{kg})(0.990)^2(3.00 \times 10^8\ \text{m/s})^2\\ &= 4.0179 \times 10^{-14}\ \text{J}.\end{aligned}$$

e. Convert units:

$$\begin{aligned} K_{\text{class}} &= 4.0179 \times 10^{-14}\ \text{J}\left(\frac{1\ \text{MeV}}{1.60 \times 10^{-13}\ \text{J}}\right) \\ &= 0.251\ \text{Mev.} \end{aligned}$$

Significance

As might be expected, because the velocity is 99.0% of the speed of light, the classical kinetic energy differs significantly from the correct relativistic value. Note also that the classical value is much smaller than the relativistic value. In fact, $K_{\text{rel}}/K_{\text{class}} = 12.4$ in this case. This illustrates how difficult it is to get a mass moving close to the speed of light. Much more energy is needed than predicted classically. Ever-increasing amounts of energy are needed to get the velocity of a mass a little closer to that of light. An energy of 3 MeV is a very small amount for an electron, and it can be achieved with present-day particle accelerators. SLAC, for example, can accelerate electrons to over $50 \times 10^9\,\text{eV} = 50{,}000\,\text{MeV}$.

Is there any point in getting *v* a little closer to *c* than 99.0% or 99.9%? The answer is yes. We learn a great deal by doing this. The energy that goes into a high-velocity mass can be converted into any other form, including into entirely new particles. In the Large Hadron Collider in **Figure 5.27**, charged particles are accelerated before entering the ring-like structure. There, two beams of particles are accelerated to their final speed of about 99.7% the speed of light in opposite directions, and made to collide, producing totally new species of particles. Most of what we know about the substructure of matter and the collection of exotic short-lived particles in nature has been learned this way. Patterns in the characteristics of these previously unknown particles hint at a basic substructure for all matter. These particles and some of their characteristics will be discussed in a later chapter on particle physics.

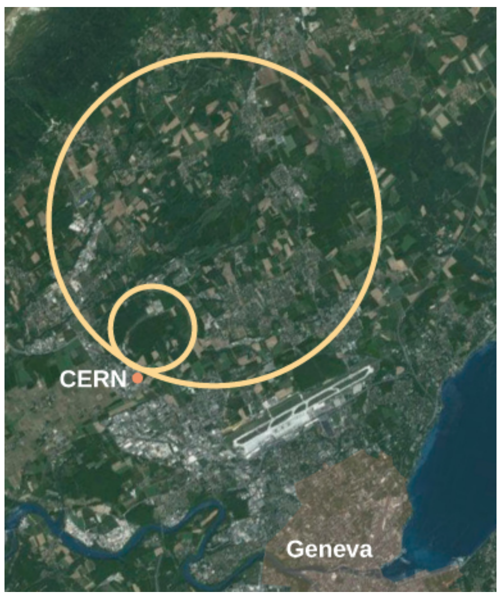


Figure 5.27 The European Organization for Nuclear Research (called CERN after its French name) operates the largest particle accelerator in the world, straddling the border between France and Switzerland. (credit: modification of work by NASA)

Total Relativistic Energy

The expression for kinetic energy can be rearranged to:

$$E = \frac{mu^2}{\sqrt{1 - u^2/c^2}} = K + mc^2.$$

Einstein argued in a separate article, also later published in 1905, that if the energy of a particle changes by ΔE, its mass changes by $\Delta m = \Delta E/c^2$. Abundant experimental evidence since then confirms that mc^2 corresponds to the energy that the particle of mass *m* has when at rest. For example, when a neutral pion of mass *m* at rest decays into two photons, the photons have zero mass but are observed to have total energy corresponding to mc^2 for the pion. Similarly, when a particle of mass m decays into two or more particles with smaller total mass, the observed kinetic energy imparted to the products of the decay corresponds to the decrease in mass. Thus, E is the total relativistic energy of the particle, and mc^2 is its rest energy.

Total Energy

Total energy E of a particle is

$$E = \gamma mc^2 \tag{5.9}$$

where m is mass, c is the speed of light, $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$, and u is the velocity of the mass relative to an observer.

Rest Energy

Rest energy of an object is

$$E_0 = mc^2. \tag{5.10}$$

This is the correct form of Einstein's most famous equation, which for the first time showed that energy is related to the mass of an object at rest. For example, if energy is stored in the object, its rest mass increases. This also implies that mass can be destroyed to release energy. The implications of these first two equations regarding relativistic energy are so broad that they were not completely recognized for some years after Einstein published them in 1905, nor was the experimental proof that they are correct widely recognized at first. Einstein, it should be noted, did understand and describe the meanings and implications of his theory.

Example 5.13

Calculating Rest Energy

Calculate the rest energy of a 1.00-g mass.

Strategy

One gram is a small mass—less than one-half the mass of a penny. We can multiply this mass, in SI units, by the speed of light squared to find the equivalent rest energy.

Solution

a. Identify the knowns: $m = 1.00 \times 10^{-3}\,\text{kg}$; $c = 3.00 \times 10^8\,\text{m/s}$.

b. Identify the unknown: E_0.

c. Express the answer as an equation: $E_0 = mc^2$.

d. Do the calculation:

$$\begin{aligned} E_0 &= mc^2 = (1.00 \times 10^{-3}\text{ kg})(3.00 \times 10^8\text{ m/s})^2 \\ &= 9.00 \times 10^{13}\text{ kg} \cdot \text{m}^2/\text{s}^2. \end{aligned}$$

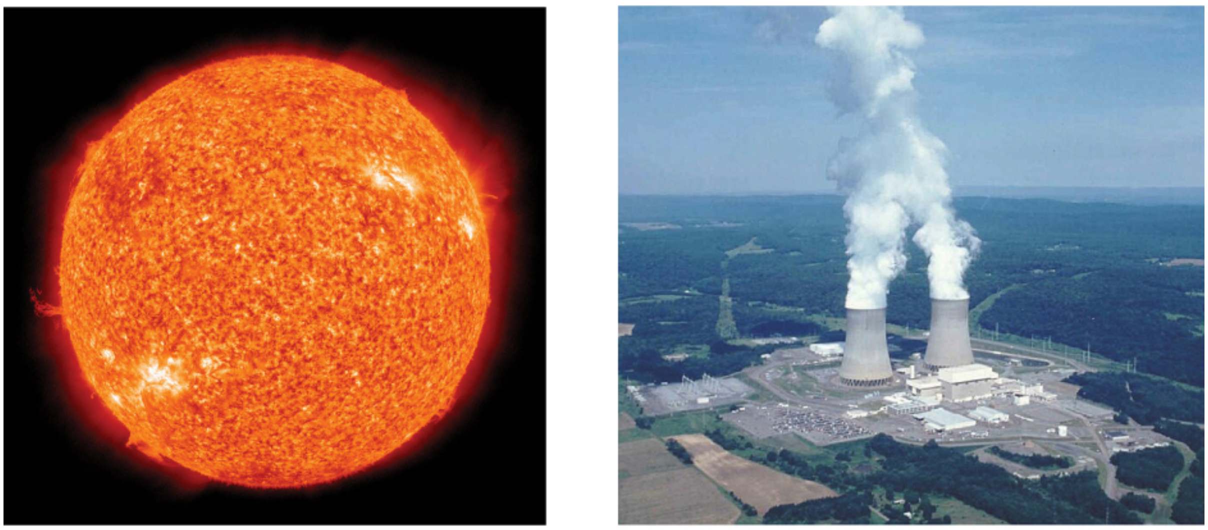
e. Convert units. Noting that $1\text{ kg} \cdot \text{m}^2/\text{s}^2 = 1\text{ J}$, we see the rest energy is:

$$E_0 = 9.00 \times 10^{13}\text{ J}.$$

Significance

This is an enormous amount of energy for a 1.00-g mass. Rest energy is large because the speed of light c is a large number and c^2 is a very large number, so that mc^2 is huge for any macroscopic mass. The $9.00 \times 10^{13}\,\mathrm{J}$ rest mass energy for 1.00 g is about twice the energy released by the Hiroshima atomic bomb and about 10,000 times the kinetic energy of a large aircraft carrier.

Today, the practical applications of *the conversion of mass into another form of energy*, such as in nuclear weapons and nuclear power plants, are well known. But examples also existed when Einstein first proposed the correct form of relativistic energy, and he did describe some of them. Nuclear radiation had been discovered in the previous decade, and it had been a mystery as to where its energy originated. The explanation was that, in some nuclear processes, a small amount of mass is destroyed and energy is released and carried by nuclear radiation. But the amount of mass destroyed is so small that it is difficult to detect that any is missing. Although Einstein proposed this as the source of energy in the radioactive salts then being studied, it was many years before there was broad recognition that mass could be and, in fact, commonly is, converted to energy (**Figure 5.28**).



(a) (b)

Figure 5.28 (a) The sun and (b) the Susquehanna Steam Electric Station both convert mass into energy—the sun via nuclear fusion, and the electric station via nuclear fission. (credit a: modification of work by NASA/SDO (AIA))

Because of the relationship of rest energy to mass, we now consider mass to be a form of energy rather than something separate. There had not been even a hint of this prior to Einstein's work. Energy-mass equivalence is now known to be the source of the sun's energy, the energy of nuclear decay, and even one of the sources of energy keeping Earth's interior hot.

Stored Energy and Potential Energy

What happens to energy stored in an object at rest, such as the energy put into a battery by charging it, or the energy stored in a toy gun's compressed spring? The energy input becomes part of the total energy of the object and thus increases its rest mass. All stored and potential energy becomes mass in a system. In seeming contradiction, the principle of conservation of mass (meaning total mass is constant) was one of the great laws verified by nineteenth-century science. Why was it not noticed to be incorrect? The following example helps answer this question.

Example 5.14

Calculating Rest Mass

A car battery is rated to be able to move 600 ampere-hours $(\mathrm{A \cdot h})$ of charge at 12.0 V. (a) Calculate the increase in rest mass of such a battery when it is taken from being fully depleted to being fully charged, assuming none of the chemical reactants enter or leave the battery. (b) What percent increase is this, given that the battery's mass is 20.0 kg?

Strategy

In part (a), we first must find the energy stored as chemical energy E_{batt} in the battery, which equals the electrical energy the battery can provide. Because $E_{\text{batt}} = qV,$ we have to calculate the charge q in $600\ \text{A} \cdot \text{h},$ which is the product of the current I and the time t. We then multiply the result by 12.0 V. We can then calculate the battery's increase in mass using $E_{\text{batt}} = (\Delta m)c^2.$ Part (b) is a simple ratio converted into a percentage.

Solution for (a)

a. Identify the knowns: $I \cdot t = 600\ \text{A} \cdot \text{h};\ V = 12.0\ \text{V};\ c = 3.00 \times 10^8\ \text{m/s}.$

b. Identify the unknown: $\Delta m.$

c. Express the answer as an equation:

$$\begin{aligned} E_{\text{batt}} &= (\Delta m)c^2 \\ \Delta m &= \frac{E_{\text{batt}}}{c^2} \\ &= \frac{qV}{c^2} \\ &= \frac{(It)V}{c^2}. \end{aligned}$$

d. Do the calculation:

$$\Delta m = \frac{(600\ \text{A} \cdot \text{h})(12.0\ \text{V})}{(3.00 \times 10^8)^2}.$$

Write amperes A as coulombs per second (C/s), and convert hours into seconds:

$$\begin{aligned} \Delta m &= \frac{(600\ \text{C/s} \cdot \text{h})\left(\frac{3600\ \text{s}}{1\ \text{h}}\right)(12.0\ \text{J/C})}{(3.00 \times 10^8\ \text{m/s})^2} \\ &= 2.88 \times 10^{-10}\ \text{kg}. \end{aligned}$$

where we have used the conversion $1\ \text{kg} \cdot \text{m}^2/\text{s}^2 = 1\ \text{J}.$

Solution for (b)

For part (b):

a. Identify the knowns: $\Delta m = 2.88 \times 10^{-10}\ \text{kg};\ m = 20.0\ \text{kg}.$

b. Identify the unknown: % change.

c. Express the answer as an equation: $\%\ \text{increase} = \frac{\Delta m}{m} \times 100\%.$

d. Do the calculation:

$$\begin{aligned} \%\ \text{increase} &= \frac{\Delta m}{m} \times 100\% \\ &= \frac{2.88 \times 10^{-10}\ \text{kg}}{20.0\ \text{kg}} \times 100\% \\ &= 1.44 \times 10^{-9}\ \%. \end{aligned}$$

Significance

Both the actual increase in mass and the percent increase are very small, because energy is divided by $c^2,$ a very large number. We would have to be able to measure the mass of the battery to a precision of a billionth of a percent, or 1 part in $10^{11},$ to notice this increase. It is no wonder that the mass variation is not readily observed.

In fact, this change in mass is so small that we may question how anyone could verify that it is real. The answer is found in nuclear processes in which the percentage of mass destroyed is large enough to be measured accurately. The mass of the fuel of a nuclear reactor, for example, is measurably smaller when its energy has been used. In that case, stored energy has been released (converted mostly into thermal energy to power electric generators) and the rest mass has decreased. A decrease in mass also occurs from using the energy stored in a battery, except that the stored energy is much greater in nuclear processes, making the change in mass measurable in practice as well as in theory.

Relativistic Energy and Momentum

We know classically that kinetic energy and momentum are related to each other, because:

$$K_{\text{class}} = \frac{p^2}{2m} = \frac{(mu)^2}{2m} = \frac{1}{2}mu^2.$$

Relativistically, we can obtain a relationship between energy and momentum by algebraically manipulating their defining equations. This yields:

$$E^2 = (pc)^2 + (mc^2)^2, \tag{5.11}$$

where E is the relativistic total energy, $E = mc^2/\sqrt{1 - u^2/c^2}$, and p is the relativistic momentum. This relationship between relativistic energy and relativistic momentum is more complicated than the classical version, but we can gain some interesting new insights by examining it. First, total energy is related to momentum and rest mass. At rest, momentum is zero, and the equation gives the total energy to be the rest energy mc^2 (so this equation is consistent with the discussion of rest energy above). However, as the mass is accelerated, its momentum p increases, thus increasing the total energy. At sufficiently high velocities, the rest energy term $(mc^2)^2$ becomes negligible compared with the momentum term $(pc)^2$; thus, $E = pc$ at extremely relativistic velocities.

If we consider momentum p to be distinct from mass, we can determine the implications of the equation $E^2 = (pc)^2 + (mc^2)^2$, for a particle that has no mass. If we take m to be zero in this equation, then $E = pc$, or $p = E/c$. Massless particles have this momentum. There are several massless particles found in nature, including photons (which are packets of electromagnetic radiation). Another implication is that a massless particle must travel at speed c and only at speed c. It is beyond the scope of this text to examine the relationship in the equation $E^2 = (pc)^2 + (mc^2)^2$ in detail, but you can see that the relationship has important implications in special relativity.

5.9 Check Your Understanding What is the kinetic energy of an electron if its speed is $0.992c$?

CHAPTER 5 REVIEW

KEY TERMS

classical (Galilean) velocity addition method of adding velocities when $v << c$; velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where *v* is the velocity between two observers, *u* is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer

event occurrence in space and time specified by its position and time coordinates (*x*, *y*, *z*, *t*) measured relative to a frame of reference

first postulate of special relativity laws of physics are the same in all inertial frames of reference

Galilean relativity if an observer measures a velocity in one frame of reference, and that frame of reference is moving with a velocity past a second reference frame, an observer in the second frame measures the original velocity as the vector sum of these velocities

Galilean transformation relation between position and time coordinates of the same events as seen in different reference frames, according to classical mechanics

inertial frame of reference reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force

length contraction decrease in observed length of an object from its proper length L_0 to length *L* when its length is observed in a reference frame where it is traveling at speed *v*

Lorentz transformation relation between position and time coordinates of the same events as seen in different reference frames, according to the special theory of relativity

Michelson-Morley experiment investigation performed in 1887 that showed that the speed of light in a vacuum is the same in all frames of reference from which it is viewed

proper length L_0; the distance between two points measured by an observer who is at rest relative to both of the points; for example, earthbound observers measure proper length when measuring the distance between two points that are stationary relative to Earth

proper time $\Delta\tau$ is the time interval measured by an observer who sees the beginning and end of the process that the time interval measures occur at the same location

relativistic kinetic energy kinetic energy of an object moving at relativistic speeds

relativistic momentum $\vec{\mathbf{p}}$, the momentum of an object moving at relativistic velocity; $\vec{\mathbf{p}} = \gamma m \vec{\mathbf{u}}$

relativistic velocity addition method of adding velocities of an object moving at a relativistic speeds

rest energy energy stored in an object at rest: $E_0 = mc^2$

rest frame frame of reference in which the observer is at rest

rest mass mass of an object as measured by an observer at rest relative to the object

second postulate of special relativity light travels in a vacuum with the same speed *c* in any direction in all inertial frames

special theory of relativity theory that Albert Einstein proposed in 1905 that assumes all the laws of physics have the same form in every inertial frame of reference, and that the speed of light is the same within all inertial frames

speed of light ultimate speed limit for any particle having mass

time dilation lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location)

total energy sum of all energies for a particle, including rest energy and kinetic energy, given for a particle of mass *m* and speed u by $E = \gamma mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

world line path through space-time

KEY EQUATIONS

Time dilation	$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \tau$
Lorentz factor	$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$
Length contraction	$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$
Galilean transformation	$x = x' + vt, \quad y = y', \quad z = z', \quad t = t'$
Lorentz transformation	$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$
	$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$
	$y = y'$
	$z = z'$
Inverse Lorentz transformation	$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$
	$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$
	$y' = y$
	$z' = z$
Space-time invariants	$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2$
	$(\Delta \tau)^2 = -(\Delta s)^2/c^2 = (\Delta t)^2 - \frac{\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]}{c^2}$
Relativistic velocity addition	$u_x = \left(\frac{u'_x + v}{1 + vu'_x/c^2}\right), \quad u_y = \left(\frac{u'_y/\gamma}{1 + vu'_x/c^2}\right), \quad u_z = \left(\frac{u'_z/\gamma}{1 + vu'_x/c^2}\right)$
Relativistic Doppler effect for wavelength	$\lambda_{obs} = \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$
Relativistic Doppler effect for frequency	$f_{obs} = f_s \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$
Relativistic momentum	$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{u}} = \frac{m \vec{\mathbf{u}}}{\sqrt{1 - \frac{u^2}{c}}}$

Relativistic total energy	$E = \gamma mc^2, \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$
Relativistic kinetic energy	$K_{\text{rel}} = (\gamma - 1)mc^2, \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

SUMMARY

5.1 Invariance of Physical Laws

- Relativity is the study of how observers in different reference frames measure the same event.
- Modern relativity is divided into two parts. Special relativity deals with observers in uniform (unaccelerated) motion, whereas general relativity includes accelerated relative motion and gravity. Modern relativity is consistent with all empirical evidence thus far and, in the limit of low velocity and weak gravitation, gives close agreement with the predictions of classical (Galilean) relativity.
- An inertial frame of reference is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.
- Modern relativity is based on Einstein's two postulates. The first postulate of special relativity is that the laws of physics are the same in all inertial frames of reference. The second postulate of special relativity is that the speed of light c is the same in all inertial frames of reference, independent of the relative motion of the observer and the light source.
- The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the sun.

5.2 Relativity of Simultaneity

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time (such as by receiving light from the events).
- Two events at locations a distance apart that are simultaneous for an observer at rest in one frame of reference are not necessarily simultaneous for an observer at rest in a different frame of reference.

5.3 Time Dilation

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time. They are not necessarily simultaneous to all observers—simultaneity is not absolute.
- Time dilation is the lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location).
- Observers moving at a relative velocity v do not measure the same elapsed time between two events. Proper time $\Delta\tau$ is the time measured in the reference frame where the start and end of the time interval occur at the same location. The time interval Δt measured by an observer who sees the frame of events moving at speed v is related to the proper time interval $\Delta\tau$ of the events by the equation:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\Delta\tau,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

- The premise of the twin paradox is faulty because the traveling twin is accelerating. The journey is not symmetrical for the two twins.

- Time dilation is usually negligible at low relative velocities, but it does occur, and it has been verified by experiment.
- The proper time is the shortest measure of any time interval. Any observer who is moving relative to the system being observed measures a time interval longer than the proper time.

5.4 Length Contraction

- All observers agree upon relative speed.
- Distance depends on an observer's motion. Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.
- Length contraction is the decrease in observed length of an object from its proper length L_0 to length L when its length is observed in a reference frame where it is traveling at speed v.
- The proper length is the longest measurement of any length interval. Any observer who is moving relative to the system being observed measures a length shorter than the proper length.

5.5 The Lorentz Transformation

- The Galilean transformation equations describe how, in classical nonrelativistic mechanics, the position, velocity, and accelerations measured in one frame appear in another. Lengths remain unchanged and a single universal time scale is assumed to apply to all inertial frames.
- Newton's laws of mechanics obey the principle of having the same form in all inertial frames under a Galilean transformation, given by

$$x = x' + vt, \quad y = y', \quad z = z', \quad t = t'.$$

  The concept that times and distances are the same in all inertial frames in the Galilean transformation, however, is inconsistent with the postulates of special relativity.
- The relativistically correct Lorentz transformation equations are

Lorentz transformation	Inverse Lorentz transformation
$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$	$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$
$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$	$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$
$y = y'$	$y' = y$
$z = z'$	$z' = z$

  We can obtain these equations by requiring an expanding spherical light signal to have the same shape and speed of growth, c, in both reference frames.
- Relativistic phenomena can be explained in terms of the geometrical properties of four-dimensional space-time, in which Lorentz transformations correspond to rotations of axes.
- The Lorentz transformation corresponds to a space-time axis rotation, similar in some ways to a rotation of space axes, but in which the invariant spatial separation is given by Δs rather than distances Δr, and that the Lorentz transformation involving the time axis does not preserve perpendicularity of axes or the scales along the axes.
- The analysis of relativistic phenomena in terms of space-time diagrams supports the conclusion that these phenomena result from properties of space and time itself, rather than from the laws of electromagnetism.

5.6 Relativistic Velocity Transformation

- With classical velocity addition, velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where v is the velocity between two observers, u is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer.

- Velocities cannot add to be greater than the speed of light.
- Relativistic velocity addition describes the velocities of an object moving at a relativistic velocity.

5.7 Doppler Effect for Light

- An observer of electromagnetic radiation sees relativistic Doppler effects if the source of the radiation is moving relative to the observer. The wavelength of the radiation is longer (called a red shift) than that emitted by the source when the source moves away from the observer and shorter (called a blue shift) when the source moves toward the observer. The shifted wavelength is described by the equation:

$$\lambda_{\text{obs}} = \lambda_s \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}.$$

where λ_{obs} is the observed wavelength, λ_s is the source wavelength, and v is the relative velocity of the source to the observer.

5.8 Relativistic Momentum

- The law of conservation of momentum is valid for relativistic momentum whenever the net external force is zero. The relativistic momentum is $p = \gamma m u$, where m is the rest mass of the object, u is its velocity relative to an observer, and the relativistic factor is $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$.
- At low velocities, relativistic momentum is equivalent to classical momentum.
- Relativistic momentum approaches infinity as u approaches c. This implies that an object with mass cannot reach the speed of light.

5.9 Relativistic Energy

- The relativistic work-energy theorem is $W_{\text{net}} = E - E_0 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$.
- Relativistically, $W_{\text{net}} = K_{\text{rel}}$ where K_{rel} is the relativistic kinetic energy.
- An object of *mass m* at velocity u has kinetic energy $K_{\text{rel}} = (\gamma - 1)mc^2$, where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$.
- At low velocities, relativistic kinetic energy reduces to classical kinetic energy.
- No object with mass can attain the speed of light, because an infinite amount of work and an infinite amount of energy input is required to accelerate a mass to the speed of light.
- Relativistic energy is conserved as long as we define it to include the possibility of mass changing to energy.
- The total energy of a particle with mass m traveling at speed u is defined as $E = \gamma mc^2$, where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ and u denotes the velocity of the particle.
- The rest energy of an object of mass m is $E_0 = mc^2$, meaning that mass is a form of energy. If energy is stored in an object, its mass increases. Mass can be destroyed to release energy.
- We do not ordinarily notice the increase or decrease in mass of an object because the change in mass is so small for a large increase in energy. The equation $E^2 = (pc)^2 + (mc^2)^2$ relates the relativistic total energy E and the relativistic momentum p. At extremely high velocities, the rest energy mc^2 becomes negligible, and $E = pc$.

CONCEPTUAL QUESTIONS

5.1 Invariance of Physical Laws

1. Which of Einstein's postulates of special relativity includes a concept that does not fit with the ideas of classical physics? Explain.

2. Is Earth an inertial frame of reference? Is the sun? Justify your response.

3. When you are flying in a commercial jet, it may appear to you that the airplane is stationary and Earth is moving beneath you. Is this point of view valid? Discuss briefly.

5.3 Time Dilation

4. (a) Does motion affect the rate of a clock as measured by an observer moving with it? (b) Does motion affect how an observer moving relative to a clock measures its rate?

5. To whom does the elapsed time for a process seem to be longer, an observer moving relative to the process or an observer moving with the process? Which observer measures the interval of proper time?

6. (a) How could you travel far into the future of Earth without aging significantly? (b) Could this method also allow you to travel into the past?

5.4 Length Contraction

7. To whom does an object seem greater in length, an observer moving with the object or an observer moving relative to the object? Which observer measures the object's proper length?

8. Relativistic effects such as time dilation and length contraction are present for cars and airplanes. Why do these effects seem strange to us?

9. Suppose an astronaut is moving relative to Earth at a significant fraction of the speed of light. (a) Does he observe the rate of his clocks to have slowed? (b) What change in the rate of earthbound clocks does he see? (c) Does his ship seem to him to shorten? (d) What about the distance between two stars that lie in the direction of his motion? (e) Do he and an earthbound observer agree on his velocity relative to Earth?

5.7 Doppler Effect for Light

10. Explain the meaning of the terms "red shift" and "blue shift" as they relate to the relativistic Doppler effect.

11. What happens to the relativistic Doppler effect when relative velocity is zero? Is this the expected result?

12. Is the relativistic Doppler effect consistent with the classical Doppler effect in the respect that λ_{obs} is larger for motion away?

13. All galaxies farther away than about 50×10^6 ly exhibit a red shift in their emitted light that is proportional to distance, with those farther and farther away having progressively greater red shifts. What does this imply, assuming that the only source of red shift is relative motion?

5.8 Relativistic Momentum

14. How does modern relativity modify the law of conservation of momentum?

15. Is it possible for an external force to be acting on a system and relativistic momentum to be conserved? Explain.

5.9 Relativistic Energy

16. How are the classical laws of conservation of energy and conservation of mass modified by modern relativity?

17. What happens to the mass of water in a pot when it cools, assuming no molecules escape or are added? Is this observable in practice? Explain.

18. Consider a thought experiment. You place an expanded balloon of air on weighing scales outside in the early morning. The balloon stays on the scales and you are able to measure changes in its mass. Does the mass of the balloon change as the day progresses? Discuss the difficulties in carrying out this experiment.

19. The mass of the fuel in a nuclear reactor decreases by an observable amount as it puts out energy. Is the same true for the coal and oxygen combined in a conventional power plant? If so, is this observable in practice for the coal and oxygen? Explain.

20. We know that the velocity of an object with mass has an upper limit of *c*. Is there an upper limit on its momentum? Its energy? Explain.

21. Given the fact that light travels at *c* , can it have mass? Explain.

22. If you use an Earth-based telescope to project a laser beam onto the moon, you can move the spot across the moon's surface at a velocity greater than the speed of light. Does this violate modern relativity? (Note that light is being sent from the Earth to the moon, not across the surface of the moon.)

PROBLEMS

5.3 Time Dilation

23. (a) What is γ if $v = 0.250c$? (b) If $v = 0.500c$?

24. (a) What is γ if $v = 0.100c$? (b) If $v = 0.900c$?

25. Particles called π-mesons are produced by accelerator beams. If these particles travel at 2.70×10^8 m/s and live 2.60×10^{-8} s when at rest relative to an observer, how long do they live as viewed in the laboratory?

26. Suppose a particle called a kaon is created by cosmic radiation striking the atmosphere. It moves by you at $0.980c$, and it lives 1.24×10^{-8} s when at rest relative to an observer. How long does it live as you observe it?

27. A neutral π-meson is a particle that can be created by accelerator beams. If one such particle lives 1.40×10^{-16} s as measured in the laboratory, and 0.840×10^{-16} s when at rest relative to an observer, what is its velocity relative to the laboratory?

28. A neutron lives 900 s when at rest relative to an observer. How fast is the neutron moving relative to an observer who measures its life span to be 2065 s?

29. If relativistic effects are to be less than 1%, then γ must be less than 1.01. At what relative velocity is $\gamma = 1.01$?

30. If relativistic effects are to be less than 3%, then γ must be less than 1.03. At what relative velocity is $\gamma = 1.03$?

5.4 Length Contraction

31. A spaceship, 200 m long as seen on board, moves by the Earth at 0.970*c*. What is its length as measured by an earthbound observer?

32. How fast would a 6.0 m-long sports car have to be going past you in order for it to appear only 5.5 m long?

33. (a) How far does the muon in **Example 5.3** travel according to the earthbound observer? (b) How far does it travel as viewed by an observer moving with it? Base your calculation on its velocity relative to the Earth and the time it lives (proper time). (c) Verify that these two distances are related through length contraction $\gamma = 3.20$.

34. (a) How long would the muon in **Example 5.3** have lived as observed on Earth if its velocity was $0.0500c$? (b) How far would it have traveled as observed on Earth? (c) What distance is this in the muon's frame?

35. **Unreasonable Results** A spaceship is heading directly toward Earth at a velocity of 0.800*c*. The astronaut on board claims that he can send a canister toward the Earth at 1.20*c* relative to Earth. (a) Calculate the velocity the canister must have relative to the spaceship. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

5.5 The Lorentz Transformation

36. Describe the following physical occurrences as events, that is, in the form (x, y, z, t): (a) A postman rings a doorbell of a house precisely at noon. (b) At the same time as the doorbell is rung, a slice of bread pops out of a toaster that is located 10 m from the door in the east direction from the door. (c) Ten seconds later, an airplane arrives at the airport, which is 10 km from the door in the east direction and 2 km to the south.

37. Describe what happens to the angle $\alpha = \tan(v/c)$, and therefore to the transformed axes in **Figure 5.17**, as the relative velocity v of the S and S' frames of reference approaches c.

38. Describe the shape of the world line on a space-time diagram of (a) an object that remains at rest at a specific position along the *x*-axis; (b) an object that moves at constant velocity *u* in the *x*-direction; (c) an object that begins at rest and accelerates at a constant rate of in the positive *x*-direction.

39. A man standing still at a train station watches two boys throwing a baseball in a moving train. Suppose the train is moving east with a constant speed of 20 m/s and one of the boys throws the ball with a speed of 5 m/s with respect to himself toward the other boy, who is 5 m west from him. What is the velocity of the ball as observed by the man on the station?

40. When observed from the sun at a particular instant, Earth and Mars appear to move in opposite directions with speeds 108,000 km/h and 86,871 km/h, respectively. What is the speed of Mars at this instant when observed from Earth?

41. A man is running on a straight road perpendicular to a train track and away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

42. A man is running on a straight road that makes $30°$ with the train track. The man is running in the direction on the road that is away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

43. In a frame at rest with respect to the billiard table, a billiard ball of mass *m* moving with speed *v* strikes another billiard ball of mass *m* at rest. The first ball comes to rest after the collision while the second ball takes off with speed *v* in the original direction of the motion of the first ball. This shows that momentum is conserved in this frame. (a) Now, describe the same collision from the perspective of a frame that is moving with speed *v* in the direction of the motion of the first ball. (b) Is the momentum conserved in this frame?

44. In a frame at rest with respect to the billiard table, two billiard balls of same mass *m* are moving toward each other with the same speed *v*. After the collision, the two balls come to rest. (a) Show that momentum is conserved in this frame. (b) Now, describe the same collision from the perspective of a frame that is moving with speed *v* in the direction of the motion of the first ball. (c) Is the momentum conserved in this frame?

45. In a frame S, two events are observed: event 1: a pion is created at rest at the origin and event 2: the pion disintegrates after time τ. Another observer in a frame S' is moving in the positive direction along the positive *x*-axis with a constant speed *v* and observes the same two events in his frame. The origins of the two frames coincide at $t = t' = 0$. (a) Find the positions and timings of these two events in the frame S' (a) according to the Galilean transformation, and (b) according to the Lorentz transformation.

5.6 Relativistic Velocity Transformation

46. If two spaceships are heading directly toward each other at 0.800*c*, at what speed must a canister be shot from the first ship to approach the other at 0.999*c* as seen by the second ship?

47. Two planets are on a collision course, heading directly toward each other at 0.250*c*. A spaceship sent from one planet approaches the second at 0.750*c* as seen by the second planet. What is the velocity of the ship relative to the first planet?

48. When a missile is shot from one spaceship toward another, it leaves the first at 0.950*c* and approaches the other at 0.750*c*. What is the relative velocity of the two ships?

49. What is the relative velocity of two spaceships if one fires a missile at the other at 0.750*c* and the other observes it to approach at 0.950*c*?

50. Prove that for any relative velocity *v* between two observers, a beam of light sent from one to the other will approach at speed *c* (provided that *v* is less than *c*, of course).

51. Show that for any relative velocity *v* between two observers, a beam of light projected by one directly away from the other will move away at the speed of light (provided that *v* is less than *c*, of course).

5.7 Doppler Effect for Light

52. A highway patrol officer uses a device that measures the speed of vehicles by bouncing radar off them and measuring the Doppler shift. The outgoing radar has a frequency of 100 GHz and the returning echo has a frequency 15.0 kHz higher. What is the velocity of the vehicle? Note that there are two Doppler shifts in echoes. Be certain not to round off until the end of the problem, because the effect is small.

5.8 Relativistic Momentum

53. Find the momentum of a helium nucleus having a mass of 6.68×10^{-27} kg that is moving at 0.200*c*.

54. What is the momentum of an electron traveling at 0.980*c*?

55. (a) Find the momentum of a 1.00×10^{9}-kg asteroid heading towards Earth at 30.0 km/s. (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

56. (a) What is the momentum of a 2000-kg satellite orbiting at 4.00 km/s? (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

57. What is the velocity of an electron that has a momentum of 3.04×10^{-21} kg · m/s ? Note that you must calculate the velocity to at least four digits to see the difference from *c*.

58. Find the velocity of a proton that has a momentum of 4.48×10^{-19} kg · m/s.

5.9 Relativistic Energy

59. What is the rest energy of an electron, given its mass is 9.11×10^{-31} kg? Give your answer in joules and MeV.

60. Find the rest energy in joules and MeV of a proton, given its mass is 1.67×10^{-27} kg.

61. If the rest energies of a proton and a neutron (the two constituents of nuclei) are 938.3 and 939.6 MeV, respectively, what is the difference in their mass in kilograms?

62. The Big Bang that began the universe is estimated to have released 10^{68} J of energy. How many stars could half this energy create, assuming the average star's mass is 4.00×10^{30} kg ?

63. A supernova explosion of a 2.00×10^{31} kg star produces 1.00×10^{44} J of energy. (a) How many kilograms of mass are converted to energy in the explosion? (b) What is the ratio $\Delta m/m$ of mass destroyed to the original mass of the star?

64. (a) Using data from **Potential Energy of a System (http://cnx.org/content/m58312/latest/#fs-id1165036086155)** , calculate the mass converted to energy by the fission of 1.00 kg of uranium. (b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

65. (a) Using data from **Potential Energy of a System (http://cnx.org/content/m58312/latest/#fs-id1165036086155)** , calculate the amount of mass converted to energy by the fusion of 1.00 kg of hydrogen. (b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$? (c) How does this compare with $\Delta m/m$ for the fission of 1.00 kg of uranium?

66. There is approximately 10^{34} J of energy available from fusion of hydrogen in the world's oceans. (a) If 10^{33} J of this energy were utilized, what would be the decrease in mass of the oceans? (b) How great a volume of water does this correspond to? (c) Comment on whether this is a significant fraction of the total mass of the oceans.

67. A muon has a rest mass energy of 105.7 MeV, and it decays into an electron and a massless particle. (a) If all the lost mass is converted into the electron's kinetic energy, find γ for the electron. (b) What is the electron's velocity?

68. A π-meson is a particle that decays into a muon and a massless particle. The π-meson has a rest mass energy of 139.6 MeV, and the muon has a rest mass energy of 105.7 MeV. Suppose the π-meson is at rest and all of the missing mass goes into the muon's kinetic energy. How fast will the muon move?

69. (a) Calculate the relativistic kinetic energy of a 1000-kg car moving at 30.0 m/s if the speed of light were only 45.0 m/s. (b) Find the ratio of the relativistic kinetic energy to classical.

70. Alpha decay is nuclear decay in which a helium nucleus is emitted. If the helium nucleus has a mass of 6.80×10^{-27} kg and is given 5.00 MeV of kinetic energy, what is its velocity?

71. (a) Beta decay is nuclear decay in which an electron is emitted. If the electron is given 0.750 MeV of kinetic energy, what is its velocity? (b) Comment on how the high velocity is consistent with the kinetic energy as it compares to the rest mass energy of the electron.

ADDITIONAL PROBLEMS

72. (a) At what relative velocity is $\gamma = 1.50$? (b) At what relative velocity is $\gamma = 100$?

73. (a) At what relative velocity is $\gamma = 2.00$? (b) At what relative velocity is $\gamma = 10.0$?

74. **Unreasonable Results** (a) Find the value of γ required for the following situation. An earthbound observer measures 23.9 h to have passed while signals from a high-velocity space probe indicate that 24.0 h have passed on board. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

75. (a) How long does it take the astronaut in **Example 5.5** to travel 4.30 ly at $0.99944c$ (as measured by the earthbound observer)? (b) How long does it take according to the astronaut? (c) Verify that these two times are related through time dilation with $\gamma = 30.00$ as given.

76. (a) How fast would an athlete need to be running for a 100- m race to look 100 yd long? (b) Is the answer consistent with the fact that relativistic effects are difficult to observe in ordinary circumstances? Explain.

77. (a) Find the value of γ for the following situation. An astronaut measures the length of his spaceship to be 100 m, while an earthbound observer measures it to be 25.0 m. (b) What is the speed of the spaceship relative to Earth?

78. A clock in a spaceship runs one-tenth the rate at which an identical clock on Earth runs. What is the speed of the spaceship?

79. An astronaut has a heartbeat rate of 66 beats per minute as measured during his physical exam on Earth. The heartbeat rate of the astronaut is measured when he is in a spaceship traveling at 0.5*c* with respect to Earth by an observer (A) in the ship and by an observer (B) on Earth. (a) Describe an experimental method by which observer B on Earth will be able to determine the heartbeat rate of the astronaut when the astronaut is in the spaceship. (b) What will be the heartbeat rate(s) of the astronaut reported by observers A and B?

80. A spaceship (A) is moving at speed *c*/2 with respect to another spaceship (B). Observers in A and B set their clocks so that the event at (*x*, *y*, *z*, *t*) of turning on a laser in spaceship B has coordinates (0, 0, 0, 0) in A and also (0, 0, 0, 0) in B. An observer at the origin of B turns on the laser at $t = 0$ and turns it off at $t = \tau$ in his time. What is the time duration between on and off as seen by an observer in A?

81. Same two observers as in the preceding exercise, but now we look at two events occurring in spaceship A. A photon arrives at the origin of A at its time $t = 0$ and another photon arrives at $(x = 1.00\,\text{m},\ 0,\ 0)$ at $t = 0$ in the frame of ship A. (a) Find the coordinates and times of the two events as seen by an observer in frame B. (b) In which frame are the two events simultaneous and in which frame are they are not simultaneous?

82. Same two observers as in the preceding exercises. A rod of length 1 m is laid out on the *x*-axis in the frame of B from origin to $(x = 1.00\,\text{m},\ 0,\ 0)$. What is the length of the rod observed by an observer in the frame of spaceship A?

83. An observer at origin of inertial frame S sees a flashbulb go off at $x = 150\,\text{km}$, $y = 15.0\,\text{km}$, and $z = 1.00\,\text{km}$ at time $t = 4.5 \times 10^{-4}\,\text{s}$. At what time and position in the S′ system did the flash occur, if S′ is moving along shared *x*-direction with S at a velocity $v = 0.6c$?

84. An observer sees two events 1.5×10^{-8} s apart at a separation of 800 m. How fast must a second observer be moving relative to the first to see the two events occur simultaneously?

85. An observer standing by the railroad tracks sees two bolts of lightning strike the ends of a 500-m-long train simultaneously at the instant the middle of the train passes him at 50 m/s. Use the Lorentz transformation to find the time between the lightning strikes as measured by a passenger seated in the middle of the train.

86. Two astronomical events are observed from Earth to occur at a time of 1 s apart and a distance separation of $1.5 \times 10^9\,\text{m}$ from each other. (a) Determine whether separation of the two events is space like or time like. (b) State what this implies about whether it is consistent with special relativity for one event to have caused the other?

87. Two astronomical events are observed from Earth to occur at a time of 0.30 s apart and a distance separation of $2.0 \times 10^9\,\text{m}$ from each other. How fast must a spacecraft travel from the site of one event toward the other to make the events occur at the same time when measured in the frame of reference of the spacecraft?

88. A spacecraft starts from being at rest at the origin and accelerates at a constant rate *g*, as seen from Earth, taken to be an inertial frame, until it reaches a speed of *c*/2. (a) Show that the increment of proper time is related to the elapsed time in Earth's frame by: $d\tau = \sqrt{1 - v^2/c^2}dt$.

(b) Find an expression for the elapsed time to reach speed *c*/2 as seen in Earth's frame. (c) Use the relationship in (a) to obtain a similar expression for the elapsed proper time to reach *c*/2 as seen in the spacecraft, and determine the ratio of the time seen from Earth with that on the spacecraft to reach the final speed.

89. (a) All but the closest galaxies are receding from our own Milky Way Galaxy. If a galaxy 12.0×10^9 ly away is receding from us at 0.900*c*, at what velocity relative to us must we send an exploratory probe to approach the other galaxy at 0.990*c* as measured from that galaxy? (b) How long will it take the probe to reach the other galaxy as measured from Earth? You may assume that the velocity of the other galaxy remains constant. (c) How long will it then take for a radio signal to be beamed back? (All of this is possible in principle, but not practical.)

90. Suppose a spaceship heading straight toward the Earth at 0.750*c* can shoot a canister at 0.500*c* relative to the ship. (a) What is the velocity of the canister relative to Earth, if it is shot directly at Earth? (b) If it is shot directly away from Earth?

91. Repeat the preceding problem with the ship heading directly away from Earth.

92. If a spaceship is approaching the Earth at 0.100*c* and a message capsule is sent toward it at 0.100*c* relative to Earth, what is the speed of the capsule relative to the ship?

93. (a) Suppose the speed of light were only 3000 m/s. A jet fighter moving toward a target on the ground at 800 m/s shoots bullets, each having a muzzle velocity of 1000 m/s. What are the bullets' velocity relative to the target? (b) If the speed of light was this small, would you observe relativistic effects in everyday life? Discuss.

94. If a galaxy moving away from the Earth has a speed of 1000 km/s and emits 656 nm light characteristic of hydrogen (the most common element in the universe). (a) What wavelength would we observe on Earth? (b) What type of electromagnetic radiation is this? (c) Why is the speed of Earth in its orbit negligible here?

95. A space probe speeding towards the nearest star moves at $0.250c$ and sends radio information at a broadcast frequency of 1.00 GHz. What frequency is received on Earth?

96. Near the center of our galaxy, hydrogen gas is moving directly away from us in its orbit about a black hole. We receive 1900 nm electromagnetic radiation and know that it was 1875 nm when emitted by the hydrogen gas. What is the speed of the gas?

97. (a) Calculate the speed of a $1.00\text{-}\mu\text{g}$ particle of dust that has the same momentum as a proton moving at 0.999*c*. (b) What does the small speed tell us about the mass of a proton compared to even a tiny amount of macroscopic matter?

98. (a) Calculate γ for a proton that has a momentum of $1.00\,\text{kg}\cdot\text{m/s}.$ (b) What is its speed? Such protons form a rare component of cosmic radiation with uncertain origins.

99. Show that the relativistic form of Newton's second law is (a) $F = m\frac{du}{dt}\frac{1}{\left(1 - u^2/c^2\right)^{3/2}}$; (b) Find the force needed to accelerate a mass of 1 kg by 1 m/s^2 when it is traveling at a velocity of *c*/2.

100. A positron is an antimatter version of the electron, having exactly the same mass. When a positron and an electron meet, they annihilate, converting all of their mass into energy. (a) Find the energy released, assuming negligible kinetic energy before the annihilation. (b) If this energy is given to a proton in the form of kinetic energy, what is its velocity? (c) If this energy is given to another electron in the form of kinetic energy, what is its velocity?

101. What is the kinetic energy in MeV of a π-meson that lives 1.40×10^{-16} s as measured in the laboratory, and 0.840×10^{-16} s when at rest relative to an observer, given that its rest energy is 135 MeV?

102. Find the kinetic energy in MeV of a neutron with a measured life span of 2065 s, given its rest energy is 939.6 MeV, and rest life span is 900s.

103. (a) Show that $(pc)^2/(mc^2)^2 = \gamma^2 - 1.$ This means that at large velocities $pc > > mc^2.$ (b) Is $E \approx pc$ when $\gamma = 30.0,$ as for the astronaut discussed in the twin paradox?

104. One cosmic ray neutron has a velocity of $0.250c$ relative to the Earth. (a) What is the neutron's total energy in MeV? (b) Find its momentum. (c) Is $E \approx pc$ in this situation? Discuss in terms of the equation given in part (a) of the previous problem.

105. What is γ for a proton having a mass energy of 938.3 MeV accelerated through an effective potential of 1.0 TV (teravolt)?

106. (a) What is the effective accelerating potential for electrons at the Stanford Linear Accelerator, if $\gamma = 1.00 \times 10^5$ for them? (b) What is their total energy (nearly the same as kinetic in this case) in GeV?

107. (a) Using data from **Potential Energy of a System (http://cnx.org/content/m58312/latest/#fs-id1165036086155)** , find the mass destroyed when the energy in a barrel of crude oil is released. (b) Given these barrels contain 200 liters and assuming the density of crude oil is $750\text{kg/m}^3,$ what is the ratio of mass destroyed to original mass, $\Delta m/m$?

108. (a) Calculate the energy released by the destruction of 1.00 kg of mass. (b) How many kilograms could be lifted to a 10.0 km height by this amount of energy?

109. A Van de Graaff accelerator utilizes a 50.0 MV potential difference to accelerate charged particles such as protons. (a) What is the velocity of a proton accelerated by such a potential? (b) An electron?

110. Suppose you use an average of $500\,\text{kW}\cdot\text{h}$ of electric energy per month in your home. (a) How long would 1.00 g of mass converted to electric energy with an efficiency of 38.0% last you? (b) How many homes could be supplied at the $500\,\text{kW}\cdot\text{h}$ per month rate for one year by the energy from the described mass conversion?

111. (a) A nuclear power plant converts energy from nuclear fission into electricity with an efficiency of 35.0%. How much mass is destroyed in one year to produce a continuous 1000 MW of electric power? (b) Do you think it would be possible to observe this mass loss if the total mass of the fuel is 10^4 kg?

112. Nuclear-powered rockets were researched for some years before safety concerns became paramount. (a) What fraction of a rocket's mass would have to be destroyed to get it into a low Earth orbit, neglecting the decrease in gravity? (Assume an orbital altitude of 250 km, and calculate both the kinetic energy (classical) and the gravitational potential energy needed.) (b) If the ship has a mass of 1.00×10^5 kg (100 tons), what total yield nuclear explosion in tons of TNT is needed?

113. The sun produces energy at a rate of 3.85×10^{26} W by the fusion of hydrogen. About 0.7% of each kilogram of hydrogen goes into the energy generated by the Sun. (a) How many kilograms of hydrogen undergo fusion each second? (b) If the sun is 90.0% hydrogen and half of this can undergo fusion before the sun changes character, how long could it produce energy at its current rate? (c) How many kilograms of mass is the sun losing per second? (d) What fraction of its mass will it have lost in the time found in part (b)?

114. Show that $E^2 - p^2c^2$ for a particle is invariant under Lorentz transformations.